Managing Decomposed Belief Functions

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Abstract

In this paper we develop a method for clustering all types of belief functions, in particular non-consonant belief functions. Such clustering is done when the belief functions concern multiple events, and all belief functions are mixed up. Clustering is performed by decomposing all belief functions into simple support and inverse simple support functions that are clustered based on their pairwise generalized weights of conflict, constrained by weights of attraction assigned to keep track of all decompositions. The generalized conflict $c \in (-\infty, \infty)$ and generalized weight of conflict $J^- \in (-\infty, \infty)$ are derived in the combination of simple support and inverse simple support functions.

1. Introduction

In earlier papers[1,5,6] we developed methods within Dempster-Shafer theory[2,10,11] to manage simple support functions (SSFs) that concern different events where the SSFs were mixed up. This was the case when it was not known a priori to which event each SSF was related. The SSFs were clustered into subsets that should be handled independently. This was based on minimizing pairwise conflicts within each cluster where conflicts served as repellence, forcing conflicting SSFs into different clusters.

This method was extended[7,8] into also handling external information of an attracting nature, where attractions between SSFs suggested they belonged together.

In this paper we develop a method for managing non-consonant belief functions concerning different events where the belief functions are mixed up[a]. This is the general case where no a priori information is available regarding which event the belief functions refer to. This method is based on the extension introducing attractions and a decomposition method for belief functions.

In short, the method can be described as first decomposing all belief functions into a set of SSFs and inverse simple support functions (ISSFs).[12] Secondly, all SSFs and ISSFs are clustered, taking account of both the conflicts between every pair of SSFs and ISSFs as well as information regarding which SSFs and ISSFs were decomposed from the same belief function.

The number of clusters in the clustering process is an input parameter that needs to be known a priori. However, determination of number of clusters is outside the scope of this paper. It can be managed with other methods, e.g. the sequential estimation method proposed by Schubert and Sidenbladh.[9]

The methodology developed in this paper is intended to manage intelligence reports whose uncertainty is represented as belief functions with several alternative nonspecific propositions, i.e. non-singleton focal elements. This can be the case when handling human intelligence (HUMINT) or for that matter sensor reports from some advanced type of sensor. Presumably, humans as information sources will also on average deliver fewer but more complex intelligence reports than simple sensor systems. Such complex intelligence or advanced sensor reports can be decomposed and managed with these methods.

For a recent overview of different alternatives to manage the combination of conflicting belief functions, see the article by Smets.[14]

We begin by describing the decomposition method for belief functions (Sec. 2). In Sec. 3 we study the characteristics of all types of combinations of SSFs and ISSFs and how generalized conflicts between SSFs and ISSFs are mapped onto weights. We demonstrate

[a] Consonant belief functions can be handled in the same way as SSFs without the method developed in this paper, by clustering the consonant belief functions without any decomposition using conflicts only.[1]

how to manage all SSFs and ISSFs using these weights together with logical constraints that keep track of the decomposition (Sec. 4). Finally, in Sec. 5, conclusions are drawn.

2. Decomposition

Definition 1. An inverse simple support function on a frame of discernment Θ is a function $m : 2^{\Theta} \rightarrow (-\infty, \infty)$ characterized by a weight $w \in (1, \infty)$ and a focal element $A \subseteq \Theta$, such that, $m(\Theta) = w, m(A) = 1 - w$ and $m(X) = 0$ when $X \notin \{A, \Theta\}$.

Let us now recall the meaning of SSFs and ISSFs:[12] An SSF $m_1(\mathrm{A}) \in [0, 1]$ represents a state of belief that "You have some reason to believe that the actual world is in A (and nothing more)". An ISSF $m_2(\mathrm{A}) \in (-\infty, 0)$ on the other hand, represents a state of belief that "You have some reason *not* to believe that the actual world is in A". Equivalently, in the terminology of[12], A_1^w where $w \in [0, 1]$ and A_2^w where $w \in (1, \infty)$, respectively. Here, w is the mass assigned to Θ in m_1 and m_2. Note the notation used, where A_1^w and A_2^w represent the support for identical subsets A of the frame given by two different SSFs or ISSFs with index number 1 and 2. The lower index is the index number of the SSF or ISSF that lends support to this subset.

The ISSF A_2^w can be understood as some reason *not* to believe in A due to its absorbing belief. A simple example is one SSF $\mathrm{A}_1^{3/4}$, i.e. $m_1(\mathrm{A}) = 1/4$ and $m_1(\Theta) = 3/4$, and one ISSF $\mathrm{A}_2^{4/3}$, i.e. $m_2(\mathrm{A}) = -1/3$ and $m_2(\Theta) = 4/3$. Combining these two functions $\mathrm{A}_1^{3/4} \oplus \mathrm{A}_2^{4/3} = \mathrm{A}_1^{3/4} \ominus \mathrm{A}_2^{3/4} = \mathrm{A}^1$ (i.e. $m_{1\oplus 2}(\Theta) = 1$) yields a vacuous belief function. Here, $\mathrm{Bel_x} \oplus \mathrm{A}_1^y = \mathrm{Bel_x} \ominus \mathrm{A}_1^{1/y}$, where $\oplus$ is Dempster's rule, and $\ominus$ is the decombination operator absorbing belief.[12] Thus, the ISSF $\mathrm{A}_2^{4/3}$ can be interpreted as $1/4$ reason *not* to believe in A, since it precisely eliminates the $1/4$ support in A expressed by $\mathrm{A}_1^{3/4}$. It means that if you previously had some $1/4$ belief in A you should now delete it. That can not be achieved by supporting the complement of A. This makes $\mathrm{A}_1^{3/4}$ and $\mathrm{A}_2^{4/3}$ into two unique components called *confidence* and *diffidence*, respectively, by Smets.[12] Now, if you start out with only one ISSF $\mathrm{A}^w, w > 1$, and nothing more, this is interpreted as if you have *no* reason to believe in A and that you need more than $1/w$ additional reason before you will start believing in it.

At precisely $1/w$ additional reason you will become completely ignorant $m(\Theta) = 1$. This is different than having some belief in A and some in A^c whose combination can never be reduced to complete ignorance.

All belief functions can be decomposed into a set of SSFs and ISSFs using the method developed by Smets.[12] The decomposition method is performed in two steps Eqs. (1) and (2). First, for any non-dogmatic belief function Bel_0, i.e. where $m_0(\Theta) > 0$, calculate the commonality number for all focal elements A by Eq. (1). We have

$$Q_0(A) = \sum_{B \supseteq A} m_0(B). \tag{1}$$

Secondly, calculate $m_i(C)$ for all decomposed SSFs and ISSFs, where $C \subseteq \Theta$ including $C = \varnothing$, and i is the ith SFF or ISSF. There will be one SSF or ISSF for each subset C of the frame unless $m_i(C)$ happens to be zero. In the general case we will have $|2^\Theta|$ SFFs and ISSFs. We get for all $C \subseteq \Theta$ including $C = \varnothing$:

$$\begin{aligned} m_i(C) &= 1 - \prod_{A \supseteq C} Q_0(A)^{(-1)^{|A|-|C|+1}} \\ m_i(\Theta) &= 1 - m_i(C). \end{aligned} \tag{2}$$

For dogmatic belief functions assign $m_0(\Theta) = \epsilon > 0$ and discount all other focal elements proportionally.

For fast computation, take the logarithm of the product terms in Eq. (2) and use the Fast Möbius Transform.[3]

3. Combining Simple Support Functions and Inverse Simple Support Functions

When combining two decomposed parts from two different belief function we face three different situations: the combination of two SSFs, one SSF and one ISSF, or two ISSFs. These situations are studied below.

3.1. *Two SSFs*

In this situation we have two SSFs where $m_1(\text{A}) \in [0, 1]$ and $m_2(\text{B}) \in [0, 1]$. When the two simple support functions are combined we receive a conflict $c_{12} \in [0, 1]$ whenever $\text{A} \cap \text{B} = \varnothing$. A weight of conflict is calculated by

$$J_{ij}^{-} = -\log(1 - c_{ij}) \tag{3}$$

and $J_{ij}^{-} \in [0, \infty)$ but will be constrained to $J_{ij}^{-} \in [0, 5]$ in our neural clustering process[1,8] for computational reasons. This will ensure convergence. The weight J_{ij}^{-} will work as repellence between m_i and m_j in the clustering process. We use the notation J_{ij}^{-} for a weight of conflict to differentiate it from J_{ij}^{+}, a weight of attraction that will be introduced in Sec. 4.

This is the usual situation. It is proper that two propositions referring to different conflicting hypotheses are not combined when they are highly conflicting. Using the conflict we obtain such a graded measure (see Ref. 5).

3.2. *One SSF and one ISSF*

The situation when combining one SSF m_1 with one ISSF m_2 is interesting and unproblematic. Here, we have A_1^w where $w \in [0, 1]$ as usual, and B_2^w where $w \in (1, \infty)$, i.e. in terms of mass functions $m_2(\mathrm{B}) \in (-\infty, 0)$.

Thus, when we combine a SSF A_1^w with an ISSF B_2^w we receive a generalized conflict $c_{12} \in (-\infty, 0]$ whenever $\mathrm{A} \cap \mathrm{B} = \varnothing$. Using Eq. (3) we get a generalized weight of conflict $J_{12}^{-} \in (-\infty, 0]$ which will serve as a weak attraction between m_1 and m_2. As before we will constrain the generalized weight of conflict for computational reasons, here to $J_{ij}^{-} \in [-5, 0]$.

The weak attraction is proper and rather immediate. If you believe in a proposition A $(\mathrm{A}_1^w, 0 \leq w \leq 1)$ and you receive further evidence indicating you have some reason *not* to believe in B $(\mathrm{B}_1^w, w > 1)$, $\mathrm{A} \cap \mathrm{B} = \varnothing$, that is an indirect weak support of A as some alternatives of the frame not supported by m_1 are disbelieved.

A simple example will demonstrate this. Suppose you have an SSF $\mathrm{A}_1^{1/2}$ and an ISSF $\mathrm{B}_2^{3/2}$ such that $\mathrm{A} \cap \mathrm{B} = \varnothing$. Combining them will result in a new type of object, henceforth called a *pseudo belief function*.[12]

In standard notation $\mathrm{A}_1^{1/2}$ is

$$m_1(\mathrm{X}) = \begin{cases} 1/2, \mathrm{X} = \mathrm{A} \\ 1/2, \mathrm{X} = \Theta \end{cases}, \tag{4}$$

and $B_2^{3/2}$ is

$$m_2(X) = \begin{cases} -1/2, X = B \\ \ \ 3/2, X = \Theta \end{cases}. \tag{5}$$

A straightforward combination of m_1 and m_2 yields a pseudo belief function

$$m_{1\oplus 2}(X) = \begin{cases} \ \ 3/4, X = A \\ -1/4, X = B \\ \ \ 3/4, X = \Theta \\ -1/4, X = \varnothing \end{cases}, \tag{6}$$

without normalization and

$$m_{1\oplus 2}(X) = \begin{cases} \ \ 3/5, X = A \\ -1/5, X = B \\ \ \ 3/5, X = \Theta \end{cases} \tag{7}$$

after normalization. This is an increase of m_1's support for A from 1/2 to 3/4 and 3/5, respectively, after combination with m_2. Note the interesting effect of normalization. Usually mass on the empty set is distributed proportionally among all focal elements by weighting up the support of the focal elements through normalization. When $m(\varnothing) < 0$, then instead the support for each focal element is weighted down to distribute support to the empty set so as to make $m(\varnothing) = 0$.

This support for the focal elements of $m_{1\oplus 2}$ is different from the one we would have if we instead had received support for B^c of 1/2, $A \cap B = \varnothing$. Assume we have

$$m_3(X) = \begin{cases} 1/2, X = B^c \\ 1/2, X = \Theta \end{cases}, \tag{8}$$

then combining m_1 and m_3 yields

$$m_{1\oplus 3}(X) = \begin{cases} 1/2, X = A \\ 1/4, X = B^c \\ 1/4, X = \Theta \end{cases}, \tag{9}$$

i.e. support for A of 1/2, or 3/4 if $B^c \equiv A$.

When two conflicting belief functions are decomposed, each into several SSFs and ISSFs, the total conflict for all pairs of two SSFs originating from different belief functions will be higher than that between the two belief functions. This is because the SSFs have higher masses on their focal elements than the corresponding belief function, now that we also have ISSFs with negative mass.

A simple example will demonstrate the situation. Let us assume two belief functions m_{a} and m_{b} whose basic belief assignments are

$$m_{\mathrm{a}}(\mathrm{X}) = \begin{cases} 1/2, \mathrm{X} = \{x, y\} \\ 3/10, \mathrm{X} = \{x, z\} \\ 1/5, \mathrm{X} = \Theta \end{cases}, \tag{10}$$

and

$$m_{\mathrm{b}}(\mathrm{X}) = \begin{cases} 1/2, \mathrm{X} = \{x, y\} \\ 3/10, \mathrm{X} = \{y, q\} \\ 1/5, \mathrm{X} = \Theta \end{cases}. \tag{11}$$

The combination of m_{a} and m_{b} yields a conflict in the intersection of each function's second focal element $\{x, z\} \cap \{y, q\} = \varnothing$ of $m_{\mathrm{a} \oplus \mathrm{b}}(\varnothing) = 9/100$.

Using the decomposition algorithm, m_{a} can be decomposed into three functions. We get two SSFs $\{x, y\}_{\mathrm{a}_1}^{2/7}$ and $\{x, z\}_{\mathrm{a}_2}^{2/5}$, and one ISSF $\{x\}_{\mathrm{a}_3}^{7/4}$, where $m_{\mathrm{a}_1} \oplus m_{\mathrm{a}_2} \oplus m_{\mathrm{a}_3} = m_{\mathrm{a}}$.

Similarly, m_{b} can be decomposed into two SSFs $\{x, y\}_{\mathrm{b}_1}^{2/7}$ and $\{y, q\}_{\mathrm{b}_2}^{2/5}$, and one ISSF $\{y\}_{\mathrm{b}_3}^{7/4}$. Of the four pairs of SSFs (one from each decomposed belief function) only m_{a_2} and m_{b_2} are in conflict; $\{x, z\} \cap \{y, q\} = \varnothing$, see Fig. 1.

Combining m_{a_2} and m_{b_2} (or for that matter all four SSFs $m_{\mathrm{a}_1}, m_{\mathrm{a}_2}, m_{\mathrm{b}_1}$ and m_{b_2}) yields a conflict $m_{\mathrm{a}_2 \oplus \mathrm{b}_2}(\varnothing) = 9/25$, i.e. four times as much conflict as in the combination $m_{\mathrm{a}} \oplus m_{\mathrm{b}}$. This will be

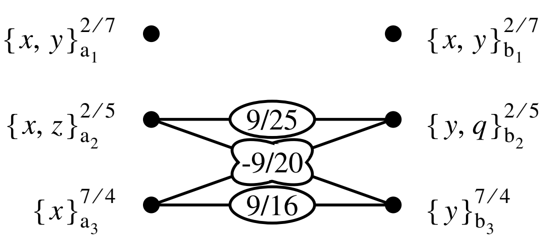

Fig. 1. Generalized conflicts between SSFs and ISSFs originating from m_{a} and m_{b}.

compensated for by a negative generalized conflict when including the two ISSFs m_{a_3} and m_{b_3} in the picture. We observe (in Fig. 1) generalized conflicts between m_{a_2} and m_{b_3}, and between m_{a_3} and m_{b_2}, respectively, i.e. $m_{a_2 \oplus b_3}(\varnothing) = m_{a_3 \oplus b_2}(\varnothing) = -9/20$.

3.3. *Two ISSFs*

The situation when combining two inverse simple support functions (ISSFs) m_1 and m_2 is perhaps the most interesting case. Here, we have two ISSFs A_1^w and B_2^w where $w \in (1, \infty)$, i.e. in terms of mass functions $m_1(A) \in (-\infty, 0)$ and $m_2(B) \in (-\infty, 0)$.

Assuming $A \cap B = \varnothing$, we receive a generalized conflict $c_{12} \in (0, \infty)$ when combining m_1 and m_2 that will serve as a repellence. This is proper but perhaps not immediately intuitive. Let us again look at an example. Let us combine $A_1^{3/2}$ and $B_2^{3/2}$, i.e. $m_1(A) = m_2(B) = -1/2$ or in other terms you have some (1/3) reason *not* to believe that the actual world is A and B, respectively, since $\mathrm{Bel}_x \oplus A_1^{3/2} = \mathrm{Bel}_x \ominus A_1^{2/3}$, where $\ominus$ is the decombination operator.[12] We have

$$m_1(X) = \begin{cases} -1/2, X = A \\ \ \ 3/2, X = \Theta \end{cases} \tag{12}$$

and

$$m_2(X) = \begin{cases} -1/2, X = B \\ \ \ 3/2, X = \Theta \end{cases} . \tag{13}$$

Combining m_1 and m_2 gives us

$$m_{1\oplus 2}(X) = \begin{cases} -3/4, X = A \\ -3/4, X = B \\ \ \ 9/4, X = \Theta \\ \ \ 1/4, X = \varnothing \end{cases} \tag{14}$$

without normalization and

$$m_{1\oplus 2}(X) = \begin{cases} -1, X = A \\ -1, X = B \\ \ \ 3, X = \Theta \end{cases} \tag{15}$$

after normalization.

The positive conflict $c_{12} = 1/4$ will serve to repel m_1 and m_2 which is proper since m_1 and m_2 contradict each other. This is observed in the decrease of belief in X = A and X = B where $m_{1\oplus 2}(\mathrm{A}) < m_1(\mathrm{A})$ and $m_{1\oplus 2}(\mathrm{B}) < m_2(\mathrm{B})$, i.e. the reason to doubt that X = A increases.

When the generalized conflict is greater than 1 we cannot use Eq. (3) to calculate a generalized weight of conflict as the logarithm is not defined for values less than 0. We call this *hyper conflicting.* We note, however, that the "1" in Eq. (3) is just a way to map a mass in the [0, 1] interval to a weight in the $[0, \infty)$ interval. As there is nothing special about the "1" in Eq. (3) other than being an upper limit for a traditional conflict we can choose any other value greater than 1 to map hyper conflicts onto weights. One radical alternative would be to adjust the value to each application by choosing to map the interval $[0, \max\{c_{ij}|\forall i,j\}]$ to the interval $[0, \infty)$ in the case with two ISSFs or $(-\infty, \max\{c_{ij}|\forall i,j\}]$ to $(-\infty, \infty)$ in the general case. We could redefine Eq. (3) as

$$J_{ij}^{-} = -\log(\max\{c_{kl}|\forall k,l\} - c_{ij}). \tag{16}$$

However, we will not do so. While this would work there are some drawbacks involved in choosing such a solution. First, if the maximum value is very high compared to most other generalized conflicts, most generalized weights of conflict would be very small which would lead to a slow convergence in the clustering process. Secondly, having a generalized conflict mapped into different generalized weights of conflict depending on the application is not attractive. Thirdly, we would like to maintain consistency with clustering only SSFs where two SSFs that flatly contradict each other for a conflict of 1 also receive a weight of conflict of ∞ and nothing less.

Thus, we will map any hyper conflicting generalized conflict greater than one to a weight of ∞. For generalized conflicts less than 0 there are of course no problems. From this we may redefine Eq. (3) as

$$J_{ij}^{-} = -\log(1 - \min\{1, c_{ij}\}), \tag{17}$$

where $J_{ij}^{-} \in (-\infty, \infty)$. As before we will, however, for computational reasons restrict the generalized weight of conflict to $J_{ij}^{-} \in [-5, 5]$.

4. Clustering SSFs and ISSFs

Having decomposed all belief functions into SSFs and ISSFs we may now cluster them using the Potts spin[15] neural clustering method extended with attractions.[8]

The Potts spin problem consists of minimizing an energy function

$$E = \frac{1}{2} \sum_{i,j=1}^{N} \sum_{a=1}^{K} (J_{ij}^{-} - J_{ij}^{+}) S_{ia} S_{ja} \tag{18}$$

by changing the states of the spins S_{ia}'s, where $S_{ia} \in \{0,\ 1\}$ and $S_{ia} = 1$ means that m_i is in cluster a. N is the number of SSFs and ISSFs, K is the number of clusters and $J_{ij}^{+} \in [0, \infty)$ is a weight of attraction formally calculated as

$$J_{ij}^{+} = -\log(1 - p_{ij}), \tag{19}$$

where p_{ij} is a basic belief assignment that m_i and m_j originate from the same belief function. This model serves as a clustering method if J_{ij}^{-} is used as a penalty factor when m_i and m_j are in the same cluster.

However, if m_i and m_j originate from the same belief function we assign $c_{ij} := 0$ and an attraction $p_{ij} := 1$, otherwise $p_{ij} := 0$. To assure smooth convergence of the neural network J_{ij}^{-} is restricted to $[-5,\ 5]$, while J_{ij}^{+} is restricted to $\{0,\ 5\}$ in this application.

Let us calculate the generalized weight of conflict between m_i and m_j, taking the restriction into account as

$$J_{ij}^{-} = \begin{cases} 0, & \exists x. m_i, m_j \in \mathrm{Bel}_x \\ -5, & \forall\, x. m_i, m_j \notin \mathrm{Bel}_x \\ & c_{ij} \leq 1 - e^{5} \\ -\ln(1 - c_{ij}), & \forall\, x. m_i, m_j \notin \mathrm{Bel}_x, \\ & 1 - e^{5} < c_{ij} < 1 - e^{-5} \\ 5, & \forall\, x. m_i, m_j \notin \mathrm{Bel}_x, \\ & c_{ij} \geq 1 - e^{-5} \end{cases}, \tag{20}$$

and assign weights of attraction as

$$J_{ij}^{+} = \begin{cases} 5, & \exists x. m_i, m_j \in \mathrm{Bel}_x \\ 0, & \forall\, x. m_i, m_j \notin \mathrm{Bel}_x \end{cases}, \tag{21}$$

enforcing the constraint that SSFs and ISSFs originating from the same belief function end up in the same cluster.

The clustering of all SSFs and ISSFs is made using the Potts spin neural clustering method extended with attractions. The minimization of the energy function, Eq. (18), is carried out by simulated annealing. In simulated annealing temperature is an important parameter. The process starts at a high temperature where the S_{ia} change state more or less at random taking little account of the interactions (J_{ij}'s). The process continues by gradually lowering the temperature. As the temperature is lowered the random flipping of spins gradually comes to a halt and the spins gradually become more influenced by the interactions (J_{ij}'s) so that a minimum of the energy function is reached. This gives us the best partition of all evidence into the clusters with minimal overall conflict.

For computational reasons we use a mean field model in order to find the minimum of the energy function. Here, spins are deterministic with $V_{ia} = \langle S_{ia} \rangle \in [0, 1]$ where V_{ia} is the expectation value of S_{ia}. The Potts mean field equations are formulated[4] as

$$V_{ia} = \frac{e^{-H_{ia}[V]/T}}{\sum_{b=1}^{K} e^{-H_{ib}[V]/T}}, \tag{22}$$

where

$$H_{ia}[V] = \sum_{j=1}^{N} (J_{ij}^{-} - J_{ij}^{+}) V_{ja} - \gamma V_{ia}, \tag{23}$$

and T is a temperature variable that is initialized to the critical temperature T_c and then lowered step-by-step during the clustering process. We have $T_c = (1/K) \cdot \max(-\lambda_{\min}, \lambda_{\max})$, where $\lambda_{\min}$ and $\lambda_{\max}$ are the extreme eigenvalues of M, where $M_{ij} = J_{ij}^{-} - J_{ij}^{+} - \gamma \delta_{ij}$.

In order to minimize the energy function Eqs. (22) and (23) are iterated until a stationary equilibrium state has been reached for each temperature. Then, the temperature is lowered step-by-step by a constant factor until $\forall i, a.\ V_{ia} \in \{0, 1\}$ in the stationary equilibrium state.[8]

5. Conclusions

In this paper we have developed a methodology which makes it possible to cluster belief functions that are mixed up. The belief functions are first decomposed into simple support functions and inverse simple support functions. We then adopt a neural clustering algorithm intended for simple support functions to handle both SSFs and ISSFs while recording their decomposition for postclustering recomposing. With this method we may cluster any type of belief function, and in particular non-consonant belief functions.

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