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Scientific report

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A review of methods of analysis for problems of penetration, field theory and constitutive equations

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Contents

1 INTRODUCTION

The use of conventional weapons against fortifications gives rise to a combination of loading from air or ground chock-waves, projectiles, fragments and shaped charges. This is literature review of these two areas together with methods on how to solve the related physical problems, i.e. how to describe material behaviour and structural response from conventional weapons loading.

2 FORTIFICATIONS AND CONVENTIONAL WEAPONS

2.1 Fortifications

Fortification (lat. forti´fico, "to strengthen") is defined as "any work erected to strengthen a position against attack" [1]. The oldest known fortification is Jeriko, whose massive stonewalls can be dated back to 8350–7550 BC. In Sweden, one can still find ancient remains of forts dating back to 300-400 AD. Until the introduction of the modern high explosive shell in the middle of the 19th century the construction materials were mainly stone, wood, soil and bricks. New construction materials such as concrete and steel armour were introduced to cope with the new threat. During the Second World War, two new threats were introduced: Shaped charges and nuclear weapons that gave rise to new and increased loading [2]. The counter-measure taken to meet this threat was the construction of underground concrete structures in rock and soil. Fortifications are usually of two types:

• Permanent fortifications, which include elaborate forts and troop shelters, are most often erected in times of peace or upon threat of war.

• Field fortifications, which are constructed when in contact with an enemy or when contact is imminent, consist of entrenched positions for personnel and crew-served weapons, cleared fields of fire, and obstacles such as explosive mines, barbed-wire entanglements, felled trees, and antitank ditches.

In 1996, new tasks were laid down on the Swedish Armed Forces and the old invasion defence is for the moment being reduced. However, several permanent fortifications will be kept in Swedish territory and they need to be upgraded and assessed when new threats arise. Among the new tasks are also international operations, which demand new types of mobile and easy assembled fortifications that will protect troops operating in foreign countries.

2.2 Effects of conventional weapons

The effects of weapons are subdivided into conventional, nuclear, biological, and chemical (NBC) weapons. Conventional weapons are commonly defined [3] as

• Projectiles without explosive charge

• Projectiles with explosive charge

• Warheads with explosively formed projectiles or shaped charges

• Warheads with other charge giving rise to effects from chock waves, pressure waves or fire

These different types of conventional weapons give rise to one or a combination of the following loading on fortifications.

• Impact of projectiles

• Impact of fragments

• Detonation of contact charges

• Air blast

• Ground chock waves

Impact of projectiles gives rise to either perforation or penetration. Perforation means that the projectile passes through the target and exits with a residual velocity. In the case of penetration, the projectile reaches a depth of penetration and either bounces back or comes to rest in the target.

2.3 Counter-measures

Three main counter-measures can be taken to increase the level of protection of a fortification:

• Conceptual (facility layout, camouflage and dummies)

• Technical (sensor activated protection, signature adaptation techniques etc.)

• Structural (strengthening)

3 SOLVING PROBLEMS OF PENETRATION

There are mainly three ways to solve problems of penetration.

3.1 Empiricism

Tests have been used for a long time to assess the level of protection of materials and structures. Various publications exist where test results are compiled in order to give fast access to empirical estimations. The compilations are either presented as graphs, for example as in [3] and [4], as empirical equations (KENNEDY 1976, bidrag M32 ISIE4) or used to create empirical databases such as DABASK [5] at FOI. During tests, registrations can be done of the following:

• The depth of penetration from post-test measurement.

• Visual captures of the penetration process using video camera or high-speed camera.

• The projectile's velocity time history from the gun muzzle to target impact using Doppler radar. In the case of a non-metallic target, part of the projectile's velocity time history during the penetration process can also be registered.

• The impact and residual velocity of the projectile from high-speed photos.

• The acceleration time history of the projectile through instrumentation of the projectile.

• Uniaxial stress time history at discrete points in the target using coal resistance gauges.

Disregarding costs, this is still the best way to asses the level of protection of a structure and both the designer and the military personnel intended to use the protective structure think of the test result as reliable.

The disadvantages are that costs are very high so that few test set-ups are tested, you have only one shot during which everything has to function correctly and often the set-up is scaled down or simplified. Another disadvantage is that only global results are given (such as depth of penetration or residual velocity) and that the process can not be resolved and studied in space or in time.

3.2 Analysis

Analysis gives more possibilities for studying the penetration process. The starting point is in field theory, described in Section 4, but then simplifications of the problem are made to be able to solve it analytically. The simplifications are based on observations from tests, or numerical simulations, where different phases and mechanisms in the penetration process are identified. Examples are models based on the cavity expansion theory, applicable at low impact velocities and for rigid projectiles, and hydrodynamic theories, where the impact velocity is so high that both target and projectile behave as fluids. A review on analytical penetration models can be found in [6]. Such analytical methods have been implemented in various codes, for example PENCURV (formerly called PENCO2D) [7] developed in the beginning of the eighties at the U.S. Army Engineer Waterways Experiment Station (WES).

The advantages of this method are that a solution relatively quickly is obtained and that parameter analysis can easily be performed.

The disadvantages are that the problem is only solved locally in the sense that no information is given on the structural level. Another disadvantage is that the assumptions made to simplify the problem, enough to solve it analytically, imposes restrictions on the range of validity of the model, as for example to rigid projectiles and semi-infinite targets.

3.3 Numerical analysis

When turning to field theories and continuum mechanics, which are described in Section 4, a system of coupled non-linear partial differential equations, called equations of balance or conservation, represents the initial-boundary value problem of penetration. By making simplifications, as described in Section 3.2, they can be solved analytically but to solve the full system we generally have to use numerical analysis where the goal is to efficiently compute accurate approximations to the solution.

Among the advantages with using numerical methods for problems of penetration, and especially with the finite element method, are that it is valid for arbitrary geometry. Also, the problem can be resolved and studied in space and time and different test set-ups can be assessed before performing expensive test.

Among the disadvantages, we find the large computational costs, high demands on the model input and the fact that the solution is an approximation.

There are four main forms of approximation.

Finite differences

This is the oldest numerical method that was first applied by Euler in 1768. In the finite difference method (FDM) the problem considered is discretized and at each grid point the derivatives are approximated by finite difference quotients. The result is a finite difference equation. This method requires a high degree of regularity and structure on the discretization and is therefor not valid for an arbitrary geometry [8].

Finite elements

The finite element method was outlined in 1943 by the mathematician Richard Courant and put to practical use on computers in the mid-1950s [1]. Due to its generality it is the most common numerical method used in continuum mechanics. The problem is spatially discretized into finite elements for which the function to be solved is approximated by test functions built up by lagrangian polynomials, shape functions. With the approximated function a weighted residual is formulated and integrated over the body, also called a weak formulation. Depending on the form of the weight function, it is called either a Bubnov-Galerkin or a Petrov-Galerkin method [9]. In the Bubnov-Galerkin method, the weight functions are constructed from the same shape functions as for the test functions and in the Petrov-Galerkin method, the shape functions for the weight are different from those used for test functions. The resulting equations from finite element approximations are called semi-discrete, since they are only discrete in space and not in time. The solution is commonly advanced in time using single or multi-step algorithms based on finite differences. If the algorithms includes solving the global equation system they are said to be implicit and if not they are explicit.

In fluid mechanics the finite volume method (FVM) introduced in the early seventies is often used. This is a Petrov-Galerkin method where the residual is weighted with point or domain collocation and the divergence theorem is applied to the result. In this method, the solution is given as an average over the elements [8]. The formulation of the finite volume method can also be constructed using finite differences. In solid mechanics, the displacements generally need to be associated with a node and the stresses with the finite element. The solution from the finite volume method then has to be projected, or interpolated, accordingly.

Using the Bubnov-Galerkin method and carrying out partial integration on the weak formulation is commonly referred to as the finite element method (FEM) and is mainly used in solid mechanics [9]. In a finite element analysis of penetration and perforation performed by the author, material description of the motion (see section 4.2) was used for both the target and the projectile together with numerical erosion [10]. It was concluded in the study that this is not a suitable way of solving the problem since the solution is strongly influenced on the erosion criteria. Instead, the motion of the target should be described in spatial co-ordinates and the motion of the projectile in material co-ordinates. This way there will be no need for numerical erosion. However, for reinforced structures this method is not possible because commercial finite element codes, at least those known to the author, lack algorithms for the concrete-reinforcement interaction.

Boundary elements

In the boundary element method (BEM) volume integrals are transformed into surface integrals. In this way, only the boundary has to be spatially discretized. The boundary element method is frequently used in geo mechanics and is mainly applicable to linear elliptic partial differential equations, which for example does not include inelastic material behaviour [11].

Meshless methods

There exist another type of methods of discretization, the so-called meshless methods. Many of them are still under development and they have not been investigated by the author. However, some of them are mentioned in order to complete the list of available tools for numerical analysis.

- Smooth particles hydrodynamics (SPH) Kernel method, introduced in 1977 in astrophysics.
- Moving least square approximation (MLS), first used in 1992.
- Reproducing Kernel methods
- Hp cloud method
- Partition of unity finite element method (PUFEM)

In the smooth particles hydrodynamics Kernel method collocation is used, i.e. it is a Petrov-Galerkin method. The other methods are Bubnov-Galerkin methods. For a detailed description of these methods see [12].

4 FIELD THEORY FOR CLASSICAL MECHANICS AND THERMODYNAMICS

The main references for this section are [13] and [14], if not otherwise stated.

4.1 Introduction

In physics there are two viewpoints in the modelling of nature of matter; corpuscular and field theory. Corpuscular theories describe matter as discontinuous composed of the smallest elements known (for the time being strings), i.e. elements that are not divisible. In field theories continuous fields represent matter, motion, energy etc. so that they are indefinitely divisible. Theories expressed in terms of field theories are called phenomenological and they are constructed from the concepts of mass, motion, stress, energy, entropy and electromagnetism. To relate these quantities field equations (also called equations of conservation or balance) are laid down. For surfaces that are singular or discontinuous with respect to a physical quantity, such as chock waves, jump conditions are introduced.

Continuum (lat. conti´nuum, from conti´nuus "connected together", "continuous", "uninterrupted") mechanics, the part of continuum physics concerned with mechanics, is a phenomenological theory for deformable media confined to Euclidean three-dimensional space. It is governed by the principle of continuity and the mechanical principles presented by Newton (Figure 4-1) in 1687, i.e. classical mechanics. The first and second law of thermodynamics often complements these principles and we then speak of thermo mechanics. The lower spatial limit of validity for continuum mechanics, based on the equations of balance and conservation, is approximately 10μm [15]. From the temporal point of view, errors of order $(u/c)^2$ appear in Newtonian mechanics where u is the relative speed and c is the speed of light [16]. In defence and space applications particle velocities of up to 10^4m/s occur and the error generated would then be $O(10^{-9})$. Euler (Figure 4-2) stated his equations of balance in 1769, which were generalised in terms of stress components by Cauchy (Figure 4-3) in 1822 [17]. Continuum mechanics is subdivided into two main areas: Kinematics that describes motion and deformation of a body and dynamics that is the study of relations between loading and resulting deformations.

Figure 4-1 Sir Isaac Newton, 1642-1727

Figure 4-2 Leonard Euler, 1707-1783

Figure 4-3 Augustin-Louis Cauchy, 1789-1857

4.2 Kinematics

4.2.1 Local motion

An event is defined in physics as a pair {**x**,t} where **x** is the spatial position vector and t the time. Under an arbitrary change of frame, comprising a rigid body motion and a time-shift, according to

$$\mathbf{x}^*=\mathbf{c}(t)+\mathbf{Q}(t)\mathbf{x},\ t^*=t-a \qquad (4.1)$$

point-distances and temporal order must be preserved, i.e. {**x**,t} and {**x***,t*} must be equivalent events. This principal of frame indifference lay down the following transformation laws for vector and tensor fields:

$$\mathbf{v}^*=\mathbf{Q}(t)\mathbf{v} \qquad (4.2)$$

$$\mathbf{S}^*=\mathbf{Q}(t)\mathbf{S}\mathbf{Q}^T(t) \qquad (4.3)$$

In this study, measures of motion of a body *B* are only summarised. For details on the subject see [14]. The invertible mapping of a body into a reference configuration in a three-dimensional Euclidean point space is denoted $\varkappa$(X). The material particle X is identified with its position vector **X** in this configuration also called the material position vector. The local motion, i.e. the one-parameter family of configurations, is given by the invertible mapping $\boldsymbol{\chi}(\mathbf{X},t)$. Grad, Div and grad, div represents differentiation with respect to **X** and **x** respectively. D and d denote material and spatial time derivatives respectively, that is keeping **X** or **x** constant. Relations between different configurations are shown in Figure 4-4.

Material particle... X

Material position vector .... $\mathbf{X}=\varkappa(X)$

Spatial position vector.................................. $\mathbf{x}=\boldsymbol{\chi}(\mathbf{X},t)$

Material velocity vector................................ $\mathbf{v}(\mathbf{X},t)=D\boldsymbol{\chi}(\mathbf{X},t)/Dt$

Material acceleration vector.......................... $\mathbf{a}(\mathbf{X},t)=D^2\boldsymbol{\chi}(\mathbf{X},t)/Dt^2$

Time derivative relation .... $D\psi(\mathbf{X},t)/Dt=d\psi(\mathbf{x},t)/dt+\mathbf{v}(\mathbf{x},t)\bullet \mathrm{grad}\psi(\mathbf{x},t)$

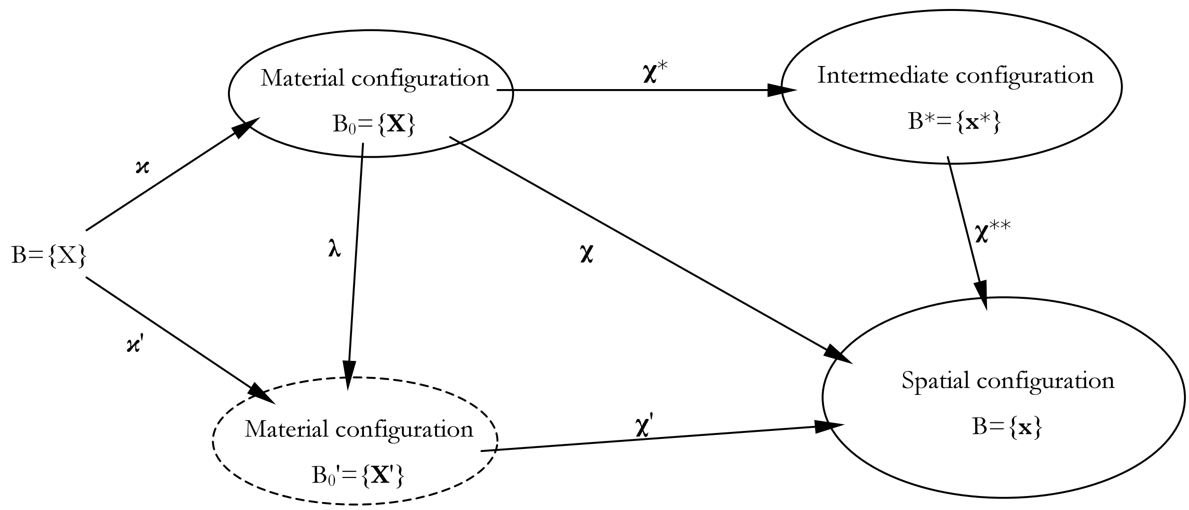

Figure 4-4 Motions and configurations

4.2.2 Local deformation

If the principle of local action is adopted, see Section 5.1, measures of deformations at a point depends only on the motion in a small neighbourhood around it. In large, or finite, deformation theory where the difference between material and spatial co-ordinates cannot be disregarded the main problem is that the strain tensors can not be additively decomposed into an elastic part and an inelastic part. There are a number of ways to circumvent this problem. One method is to split the deformation gradient multiplicatively into elastic and inelastic parts but with this method problems arise how to define elasticity [18]. Here follows a summary of measures of deformation introduced in this method.

Table 4-1 Measures of deformation.

Measure	*Material description*	*Spatial description*	*Intermediate description*
Deformation gradient	$\mathbf{F}=\mathrm{Grad}(\chi)$ $=\mathbf{F}_e\mathbf{F}_{ie}=\mathbf{RU}=\mathbf{VR}$		
Elastic deformation gradient		$\mathbf{F}_e=\partial\mathbf{x}/\partial\mathbf{x}^*$	
Inelastic deformation gradient	$\mathbf{F}_{ie}=\partial\mathbf{x}^*/\partial\mathbf{X}$		$\mathbf{F}_{ie}=\partial\mathbf{x}^*/\partial\mathbf{X}$
Rotation tensor	$\mathbf{R}$		
Right and left stretch tensor	$\mathbf{U}$	$\mathbf{V}$	
Jacobian of the deformation	$J=\det(\mathbf{F})$		
Right and left Cauchy-Green tensor	$\mathbf{C}=\mathbf{F}^T\mathbf{F}$	$\mathbf{B}=\mathbf{F}\mathbf{F}^T$	
Rate-of-deformation gradient	$D\mathbf{F}/Dt=\mathbf{LF}$		
Velocity gradient		$\mathbf{L}=\mathrm{grad}(\mathbf{v})$ $=\mathbf{L}_e+\mathbf{L}_{ie}$ $=\mathbf{l}_e+\mathbf{F}_e\mathbf{l}_{ie}\mathbf{F}_e^{-1}$	
Elastic velocity gradient		$\mathbf{l}_e=D\mathbf{F}_e/Dt\mathbf{F}_e^{-1}$	
Inelastic velocity gradient			$\mathbf{l}_{ie}=D\mathbf{F}_{ie}/Dt\mathbf{F}_{ie}^{-1}$
Rate-of-deformation tensor		$\mathbf{D}=\mathbf{D}_e+\mathbf{D}_{ie}$ $=\frac{1}{2}(\mathbf{L}+\mathbf{L}^T)$	
Spinning tensor		$\mathbf{W}=\frac{1}{2}(\mathbf{L}-\mathbf{L}^T)$	
Displacement vector		$\mathbf{u}=\mathbf{x}-\mathbf{X}$	
Displacement gradient	$\mathbf{H}=\mathrm{Grad}(\mathbf{u})$		
Material or Green-St. Venant strain tensor	$\mathbf{E}=\frac{1}{2}(\mathbf{C}-\mathbf{I})$ $=\mathbf{F}_{ie}^T\mathbf{E}_e\mathbf{F}_{ie}+\mathbf{E}_{ie}$		
Elastic material strain tensor			$\mathbf{E}_e=\frac{1}{2}(\mathbf{C}_e-\mathbf{I})$
Inelastic material strain tensor	$\mathbf{E}_{ie}=\frac{1}{2}(\mathbf{C}_{ie}-\mathbf{I})$		
Right Cauchy-Green elastic material tensor			$\mathbf{C}_e=\mathbf{F}_e^T\mathbf{F}_e$
Right Cauchy-Green inelastic material tensor	$\mathbf{C}_{ie}=\mathbf{F}_{ie}^T\mathbf{F}_{ie}$		
Spatial or Almansi strain tensor		$\mathbf{e}=\frac{1}{2}(\mathbf{I}-\mathbf{c})$	
Cauchy's deformation tensor		$\mathbf{c}=\mathbf{F}^{-T}\mathbf{F}^{-1}$	

Relations between different deformed configurations are shown in Figure 4-5 where the operations of kinematical transformations of covariant tensors are defined as:

Pull-back operation: $\mathbf{E}=\mathbf{F}^{T}\mathbf{eF}$ Push-forward operation: $\mathbf{e}=\mathbf{F}^{-T}\mathbf{EF}^{-1}$

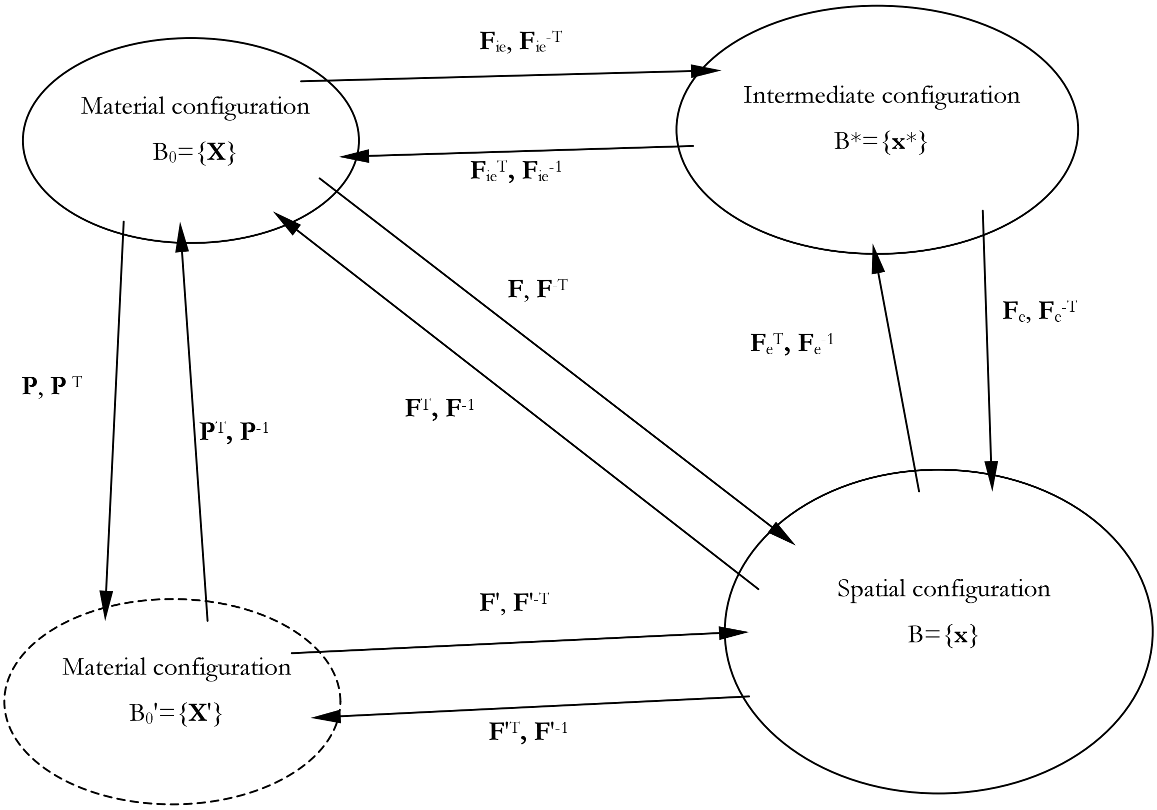

Figure 4-5 Deformations and configurations

Operations of kinetic transformations of contravariant tensors defined in material and spatial configuration are defined as:

Pull-back operation: $\mathbf{S}=\mathbf{F}^{-1}\boldsymbol{\tau}\mathbf{F}^{-T}$ Push-forward operation: $\boldsymbol{\tau}=\mathbf{FSF}^{T}$

4.2.3 Non-local deformation

If the principle of local action is not adopted the situation gets more complicated. It could be necessary to introduce non-local measures of deformation in order to avoid strain localization causing non-convergent solutions in finite element analyses [19]. A simple method to introduce non-local measures of deformation is to define an average deformation by applying a weight function to the local deformations in a domain around the point to be studied, see Figure 4-6 for an example. This weighted deformation is then used to calculate the evolution of internal variables such as plastic strain or scalar valued damage parameters. The disadvantage with this method is that another problem arises, how to choose the size of the domain. There are methods proposed on how to do this [19].

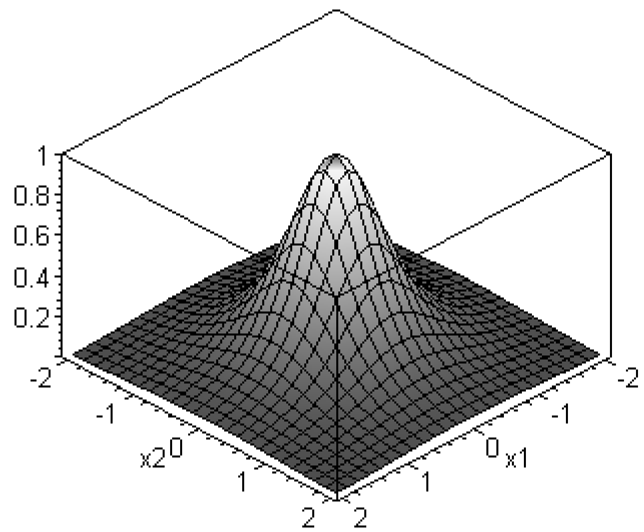

Figure 4-6 Example of a weight function for calculation of non-local deformation in a spatially discretized, two-dimensional domain. Weight function taken from [20].

4.2.4 Wave propagation

A wave is here defined as a surface of singularity s moving across a body separating it into two regions B^+ and B^-. The wave propagates from the source throughout the body, reflects and refracts when arriving at a boundary, i.e. a region with different impedance. The theory is based on Hadamard's lemma. The wave is characterised by the amplitude vector, or polarisation vector, **a**, the direction of propagation vector **n**, and the local speed of propagation U. U is the normal speed of the surface with respect to the material particles instantaneously situated upon it, that is U_s-U_p where U_s is the absolute velocity of propagation and U_p is the particle velocity.

Let a function ψ(**x**,t) be continuously defined in B^+ and B^- but not necessarily on s. The function has limit values ψ^+ and ψ^- as x approaches a point $x_0$on s, see Figure 4-7. The jump operator $[\psi]=\psi^+ - \psi^-$ denotes the jump of the discontinuous derivative at the wave front and if $[\psi]\neq 0$ the surface s is said to be singular with respect to ψ. If **a** is parallel to **n** the wave is called longitudinal, if normal transverse. For an incompressible material, only transverse waves are possible.

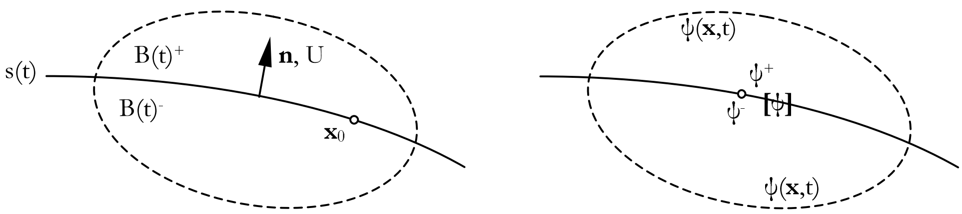

Figure 4-7 A singular surface travelling through a body.

Let the function ψ be associated with the motion χ or a derivative of the motion. If the wave propagation speeds and corresponding stretches are known, the stress relation can be derived. We characterize a singular surface with the number of continuous derivates of the motion, and define its order as the order of the first discontinuous derivative.

• Fracture is a singular surface of order zero, i.e. discontinuous motion.

$[\chi]\neq 0$

• Singular surfaces of first order, i.e. discontinuous velocity and deformation gradient.

$[\mathrm{Grad}(\chi)]\neq 0$, $[D\chi/Dt]=0$ (Material singularity)

$[\mathrm{Grad}(\chi)]\neq 0$, $[D\chi/Dt]\neq 0$, $[D\chi_t/Dt]=0$ (Chock wave)

$[\mathrm{Grad}(\chi)]\neq 0$, $[D\chi/Dt]\neq 0$, $[D\chi_n/Dt]=0$ (vortex sheet)

• Acceleration waves are singular surfaces of second order, i.e. discontinuous acceleration.

$[D^2\mathbf{x}/Dt^2]\neq 0$

Waves of higher order are defined in a similarly manner.

In finite element codes the jump conditions are not used, singular surfaces such as chock waves are treated using artificial bulk viscosity [Neumann, 1950 #115].

4.3 Equations of balance

All equations of balance are here stated as both field equations and corresponding jump condition in their local spatial form. The general equations of balance for any physical quantity ψ, **i** being the influx of ψ over the boundary, s the supply of ψ within the continuum, ϱ the mass density, **v** is the velocity vector and **n** is the direction of propagation of the wave, are:

General field equation	**General jump condition**
$\varrho D\psi/Dt+\psi[D\varrho/Dt+\varrho\mathrm{div}\mathbf{v}]-\varrho s+\mathrm{div}\mathbf{i}=0$	$[\varrho\psi U]-[\mathbf{i}]\bullet\mathbf{n}=0$

One of the basic requirements for mass is that it is invariant under motion, i.e. mass is conserved. This requirement supplies the first equation of balance given in see Table 4-2.

Table 4-2 Equation of balance from the principle of continuity.

	Field equation	*Jump condition*
mass	$D\varrho/Dt+\varrho\mathrm{div}\mathbf{v}=0$	$[\varrho U]=0$

From Newton's first law of motion, via the quantification by Euler and generalisation by Cauchy, we get the next two equations of balance, see Table 4-3, where $\boldsymbol{b}$ is the body force vector and $\boldsymbol{T}$ is the true, or Cauchy, stress tensor.

Table 4-3 Equations of balance from classical mechanics.

	Field equation	*Jump condition*
linear momentum	$\mathrm{div}\mathbf{T}^T+\varrho\mathbf{b}-\varrho\mathbf{a}=0$	$[\varrho\mathbf{v}U]+[\mathbf{T}]\bullet\mathbf{n}=\mathbf{0}$
angular momentum	$\mathbf{T}^T=\mathbf{T}$	-

From thermodynamics, the science of energy and entropy, we have the first and second law, which are both based on experimental observations. The first law asserts the conservation of energy by introducing a state variable called the internal energy ε and the assumption that the balance laws of linear momentum and angular momentum are satisfied. The second law was stated by Clausius as: "It is impossible to construct a device that operates in a cycle and produces no effect other than the transfer of heat from a cooler body to a hotter body" [21]. This law leads to the property of entropy η. The entropy production γ is always non-negative and consists of two parts [14]:

• Local entropy production which asserts that mechanical work can only be consumed and not given out by a body with uniform temperature and without heat sources.

• Conduction entropy production that asserts that heat does not flow spontaneously from colder to hotter parts of a body.

The equations of balance resulting from these two laws are shown in Table 4-4 where **h** is the heat flux vector field, q is the specific heat absorption scalar field and θ is the absolute, i.e. measured on the Kelvin scale, temperature scalar field.

Table 4-4 Equations of balance from thermodynamics.

	Field equation	*Jump condition*
energy	$\varrho D\varepsilon/Dt - tr(\mathbf{TD}) + div\mathbf{h} - \varrho q = 0$	$[\varrho(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v})U] + [\mathbf{T}\cdot\mathbf{v} - \mathbf{h}]\cdot\mathbf{n} = 0$
entropy	$\gamma_{loc} = D\eta/Dt + \varrho^{-1}\theta^{-1}div\mathbf{h} - \theta^{-1}q \geq 0$ $\gamma_{con} = -\varrho^{-1}\theta^{-2}\mathbf{h}\cdot grad\theta \geq 0$ $\gamma = \gamma_{loc} + \gamma_{con}$	$[\varrho\eta U] - [\mathbf{h}\theta^{-1}]\cdot\mathbf{n} \geq 0$

The entropy inequality is also known as the Clausius-Duhem inequality or the principle of dissipation. Combining the balance of energy and the local entropy production and introducing the specific, or Helmholtz, free energy defined as $\Psi = \varepsilon - \theta\eta$ we get an inequality that incorporates both equations of balance supplied by thermodynamics.

$$\varrho\theta\gamma = -\varrho(D\Psi/Dt + \eta D\theta/Dt) + tr(\mathbf{TD}) - \theta^{-1}\mathbf{h}grad\theta \geq 0 \qquad (4.4)$$

In numerical analysis the jump conditions are generally not used, instead singular surfaces are treated with bulk viscosity [22].

5 MECHANICAL CONSTITUTIVE EQUATIONS

The references for this section are [14] and [23] if not otherwise stated.

5.1 Introduction

Two sets {**x**,**T**} and {**x**,**T**,**b**,ε,**h**,q,η,θ,**ξ**} satisfying the laws of balance define a dynamic and a thermodynamic process, respectively. Constitutive assumptions restrict the possible processes to the admissible processes that a body can undergo, and these assumptions are necessary to formulate a determinate problem. Constitutive equations characterize the properties of ideal materials by relating the field variables, for example the stress tensor to the motion and the heat-flux vector to the temperature. Da Vinci Figure 5-1 made a sketch that has been interpreted as a simple tension test and in 1678, Robert Hooke presented his investigations on springing bodies including his law of elasticity discovered already in 1660. Euler Figure 5-2 formulated in 1769 the first constitutive equation for a material class, the non-viscous fluids [24].

Figure 5-1 Leonardo da Vinci, 1452-1519

Figure 5-2 Leonard Euler, 1707-1783

The nine axioms for constitutive equations are.

1. Consistency must prevail between the constitutive equations and the equations of balance.

2. Co-ordinate invariance means that a constitutive equation must be equally valid in all fixed co-ordinate systems. This is commonly achieved by stating the equations in tensorial form.

3. Isotropy or aeolotropy. Any material symmetry must be reflected by the constitutive equations.

4. Just setting or well posedness must be satisfied for the boundary-initial value problem, i.e. there must exist a unique solution that depends continuously on the initial and boundary value data.

5. Dimensional invariance: The constitutive equations must be fully stated, i.e. all material constants upon which the response of the material may depend must be included.

6. Material frame indifference or material objectivity: Constitutive equations must be invariant under a change of frame, i.e. independent of the observer.

7. Equipresence: All independent variables stated must be include in all constitutive equations unless stated otherwise by physical laws.

8. The principle of determinism states that the field variables are determined by the history of the motion of that body and not the future motion.

9. The principle of local action states that in determining the stress at a given particle the motion outside an arbitrary neighbourhood of the particle may be disregarded.

The principle of determinism asserts the existents of constitutive, or material response, functionals **G** such that

$$\mathbf{T}=\mathbf{G}_{\mathbf{T}}[\boldsymbol{\chi}(\mathbf{X},\tau);\mathbf{X},t]$$

$$\Psi=\mathbf{G}_{\Psi}[\boldsymbol{\chi}(\mathbf{X},\tau);\mathbf{X},t]$$

$$\eta=\mathbf{G}_{\eta}[\boldsymbol{\chi}(\mathbf{X},\tau);\mathbf{X},t]$$

$$\mathbf{h}=\mathbf{G}_{\mathbf{h}}[\boldsymbol{\chi}(\mathbf{X},\tau);\mathbf{X},t]$$

where $\tau=t-s$ and $s>=0$, that is the complete history. **G** is a functional with respect to its first argument and a function with respect to the second and third argument. For materials with internal constraints, modification of the principle of determinism is needed. The stress **T** is then determined by the history of the motion of the body only to within a stress **N** that do no work in any motion satisfying the constraints. In the case of incompressibility we have the constraint $C(\mathbf{F})=\det(\mathbf{F})-1=0$ which leads to a split of the stress tensor into a hydrostatic pressure part and a deviatoric part:

$$\mathbf{T}=\mathbf{T}'+1/3\mathrm{tr}(\mathbf{T})\mathbf{I}$$

$$\mathbf{T}'=\mathbf{G}[\boldsymbol{\chi}(\mathbf{X},\tau);\mathbf{X},t] \qquad (5.1)$$

The axiom of material frame indifference imposes the following condition on the response functional.

$$\mathbf{Q}\mathbf{G}[\boldsymbol{\chi}(\mathbf{X},\tau);\mathbf{X},t]\mathbf{Q}^T=\mathbf{G}[\mathbf{Q}\boldsymbol{\chi}(\mathbf{X},\tau);\mathbf{X},t] \qquad (5.2)$$

Adopting the principle of local action together with a Taylor expansion of the motion evaluated at the material particle X results in the response functional for a material of grade n where n is the order of the expansion.

$$\mathbf{T}(\mathbf{x},t)=\mathbf{G}[{}_1\mathbf{F}(\mathbf{X},\tau),{}_2\mathbf{F}(\mathbf{X},\tau)\ldots{}_n\mathbf{F}(\mathbf{X},\tau);\mathbf{X},t] \qquad (5.3)$$

A first order Taylor expansion of the motion evaluated at the material particle X results in the response functional for a material of grade one or, a simple material.

$$\mathbf{T}(\mathbf{x},t)=\mathbf{G}[\mathbf{F}(\mathbf{X},\tau);\mathbf{X},t] \qquad (5.4)$$

Material time derivatives of tensors appearing in continuum mechanics are often found frame-different, i.e. variant under a change of observer. One example is the stress-rate of the true, or Cauchy, stress tensor **T** used in hypoelasticity, see section 5.5. To satisfy the principle of material objectivity so called objective stress rates are used. Here follows a listing of the most commonly used objective stress-rates [18].

• Co-rotational, or Jaumann, stress rate

$$\mathbf{T}^{j}=D\mathbf{T}/Dt-\mathbf{W}\mathbf{T}-\mathbf{T}\mathbf{W}^T \qquad (5.5)$$

• Convected, or Oldroyd, stress rate

$$\mathbf{T}^{o}=\mathrm{D}\mathbf{T}/\mathrm{Dt}-\mathbf{LT}-\mathbf{TL}^{T} \quad (5.6)$$

• Green-Nagdhi stress rate

$$\mathbf{T}^{gn}=\mathrm{D}\mathbf{T}/\mathrm{Dt}-\mathrm{D}\mathbf{R}/\mathrm{Dt}\mathbf{R}^{T}\mathbf{T}-\mathbf{TR}\mathrm{D}\mathbf{R}^{T}/\mathrm{Dt} \quad (5.7)$$

5.2 Material isomorphism

If for the neighbourhoods of two material particles X and X', N(X) and N(X'), there exist two reference configurations $\varkappa$ and $\varkappa'$ such that the density is uniform and the response functional coincide according to

$$\mathbf{G}[\boldsymbol{\chi}(\mathbf{X},\tau);\mathbf{X},t]=\mathbf{G}'[\boldsymbol{\chi}(\mathbf{X}',\tau);\mathbf{X}',t^*] \quad (5.8)$$

Then the material particles are said to be materially isomorphic, i.e. the two particles consist of the same material. If all material particles are mutually isomorphic, the body is materially uniform and if the body is materially homogenous with respect to one reference configuration for all material particles, it is called materially homogeneous [25].

$$\mathbf{G}[\boldsymbol{\chi}(\mathbf{X},\tau);\mathbf{X},t]=\mathbf{G}[\boldsymbol{\chi}(\mathbf{X}',\tau);\mathbf{X}',t] \quad (5.9)$$

5.3 Material symmetry

For a change of reference configuration from $\varkappa$ to $\varkappa'$ and denoting the corresponding deformation gradient $\mathbf{P}=\mathbf{F}'^{-1}\mathbf{F}$, we get the following relationship for the (simple) material response functionals.

$$\mathbf{G}[\mathbf{F}(\mathbf{X},\tau);\mathbf{X},t]=\mathbf{G}'[\mathbf{F}(\mathbf{X},\tau)\mathbf{P}^{-1};\mathbf{X},t] \quad (5.10)$$

If the material response relatively to $\varkappa$ and $\varkappa'$ is indistinguishable the two response functionals are equal, that is

$$\mathbf{G}[\mathbf{F}(\mathbf{X},\tau);\mathbf{X},t]=\mathbf{G}[\mathbf{F}(\mathbf{X},\tau)\mathbf{P}^{-1};\mathbf{X},t] \quad (5.11)$$

We then define $\mathbf{K}=\mathbf{P}^{-1}$ as the symmetry, or isotropy, group of the material relative to $\varkappa$ that characterise the deformational symmetry of the material [25]. This symmetry group is related to the symmetry group for another reference configuration $\varkappa'$ through Noll's rule where $\mathbf{A}$ denotes the deformation gradient from $\varkappa$ to $\varkappa'$.

$$\mathbf{K}'=\mathbf{AKA}^{-1} \quad (5.12)$$

Then $\{\mathbf{K}'\}$ is the symmetry group with respect to $\varkappa'$. A special case arise if the change of reference corresponds to pure dilatation, $\mathbf{A}=p\mathbf{I}$, so that $\mathbf{K}=\mathbf{K}'$ and consequently the symmetry group of the material is unchanged.

There are two types of symmetry, geometrical and deformational. The type of deformational symmetry for a certain material and a reference configuration is concluded from the properties of this group. Isochoric linear deformation forms a subgroup of the group of linear deformations also called the unimodular group. Fluids are defined as materials whose symmetry group contains the full unimodular group

• Fluid $\{\mathbf{K} \mid \mathbf{K}\in L(\mathbf{v},\mathbf{v}),\ \det(\mathbf{K})=1\}$

The group of linear orthogonal deformations is a subgroup of the unimodular group. If the symmetry group for a material contains the full orthogonal group, it is called an isotropic solid.

- Isotropic solid $\{\mathbf{K} \mid \mathbf{K} \in L(\mathbf{v},\mathbf{v}), \det(\mathbf{K})=1, \mathbf{K}^T\mathbf{K}=\mathbf{I}\}$

If the symmetry group for a material group is a proper subgroup of the orthogonal group, it is called an aeolotropic, or anisotropic, solid.

5.4 Principal material classes

A material is said to be of differential type if its response functional depends only on the values of its arguments at times near t and on a finite number of time derivatives.

Materials of rate types are defined as those with response functional dependent not only on the values of the deformation gradient but also on the initial values on the stress tensor and its time derivatives.

Materials of integral type are defined as materials that have a response functional that can be expressed in terms of an integral polynomial over the deformation history.

5.5 Rheological material classes

- Elasticity

Elastic materials are defined as simple materials whose response functional depends on its arguments at time t only and not on its history. This is represented below in the form of Cauchy elasticity.

$$\mathbf{T}(\mathbf{x},t)=\mathbf{G}[\mathbf{F}(\mathbf{X},t);\mathbf{X},t] \qquad (5.13)$$

For materials possessing both elastic and inelastic behaviour, the initial elastic domain represented by a surface in stress space, must be convex and thus closed [26].

In **hyperelasticity**, or Green elasticity, the response functional is derived from a scalar function W representing the stored strain energy in the continuum. Hyperelastic materials belong to the material class of differential types. The true stress response functional is given by

$$\mathbf{G}[\mathbf{F}(\mathbf{X},t);\mathbf{X},t]=J^{-1}\mathbf{F}\partial W/\partial \mathbf{F} \qquad (5.14)$$

where J is the determinant of the deformation gradient.

Hypoelastic materials are defined as being of rate type and valid only for isotropic elasticity. Hypoelasticity violates the first axiom in Section 5.1 since it can be shown that it is inconsistent with the first law of thermodynamics in a closed cycle of deformation [27]. In hypoelasticity any frame-indifferent stress rate, here exemplified by the co-rotational stress rate, is given by

$$\mathbf{T}^J=\mathbf{A}:\mathbf{D} \qquad (5.15)$$

where : represents double contraction of the tensor components and **A** is an isotropic fourth order tensor with components

$A_{ijkl}=2G[\tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\nu/(1-2\nu)\delta_{ij}\delta_{kl}]$

where G is the shear modulus, ν is the ratio of transversal contraction and δ is the Kronecker's delta.

• Inelasticity

Inelastic, simple materials are defined as materials whose response functional depends on additional information compared to elastic materials. This could for example be the deformation and temperature history and complementary, or internal, variables ξ_i [28].

$$\mathbf{T}(\mathbf{x},t)=\mathbf{G}[\mathbf{F}(\mathbf{X},\tau);\mathbf{X},t,\theta,\boldsymbol{\xi}] \quad (5.16)$$

To each internal variable in the array $\boldsymbol{\xi}$, a relation governing its rate of evolution needs to be specified. For the internal variable theory, the second law of thermodynamics imposes restriction on the seventh axiom for constitutive equations, the principle of equipresence [29]. For a simple material the response functionals reads

$$\mathbf{T}= J^{-1}\mathbf{F}\ \mathbf{G}_\mathbf{S}(\mathbf{F},\theta,\boldsymbol{\xi})\mathbf{F}^T=J^{-1}\mathbf{F}\partial\mathbf{G}_\Psi(\mathbf{F},\theta,\boldsymbol{\xi})/\partial\mathbf{F}$$

$$\Psi=\mathbf{G}_\Psi(\mathbf{F},\theta,\boldsymbol{\xi})$$

$$\eta=\mathbf{G}_\eta(\mathbf{F},\theta,\boldsymbol{\xi})=-\partial\mathbf{G}_\Psi(\mathbf{F},\theta,\boldsymbol{\xi})/\partial\theta$$

$$\mathbf{h}=\mathbf{G}_\mathbf{h}(\mathbf{F},\theta,\mathrm{grad}\theta,\boldsymbol{\xi})$$

$$D\boldsymbol{\xi}/Dt=\mathbf{G}_\xi(\mathbf{F},\theta,\mathrm{grad}\theta,\boldsymbol{\xi})$$

There are a number of subclasses of inelasticity. Simplest is the theory of **viscoelasticity** where the response functional depends on the motion history through the rate of deformation. The most widely used theory of inelasticity is **plasticity**. In this theory, the material response is governed by inelastic, plastic straining whose evolution is governed by a flow rule [28]. This theory incorporates dissipative material behaviour but it is by definition only valid for infinitely slow, or quasi-static, dynamic processes. A third theory is that of **viscoplasticity** that accounts for both dissipative effects as well as the rate of the dynamic process. The combination of elasticity and viscoplasticity, where the material response is elastic up to a certain level of straining or stressing and then behaves as a viscoplastic material, is called the theory of **elasticity/viscoplasticity**. For **elastic-viscoplastic** materials, the material behaviour is viscous in both domains [26]. These two theories seem to be the rheological material classes best suited for describing material behaviour under weapons loading.

6 SUMMARY AND CONCLUSIONS

A brief review has been performed in the fields of fortifications, loading from conventional weapons, methods for solving problems of penetration, field theory for classical mechanics and thermodynamics and mechanical constitutive equations. It seems as if the following methods are best suited to solve the related problems.

- continuum mechanics to describe the problems
- numerical analysis with the Bubnov-Galerkin formulation of the finite element method
- elastic-viscoplastic constitutive equations to describe the material behaviour

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Report title

A review of methods of analysis for problems of penetration, field theory and constitutive equations.

Abstract (not more than 200 words)

A brief review has been carried out in the field of mechanical constitutive equations. First the two topics of fortifications and conventional weapons loading are explained and then a short summary of existing solution methods for problems of penetration. Field theory and thermodynamics are mentioned together with kinematics for finite elastic and inelastic deformations. The final part of the review is an introduction to mechanical constitutive equations emphasized on dynamically loaded solids.

Keywords

fortifications, conventional weapons loading, penetration, numerical analysis, field theory, constitutive equations

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En litteraturstudie inom analysmetoder för penetrationsproblem, fältteori och konstitutiva ekvationer.

Sammanfattning (högst 200 ord)

En kort litteraturstudie har genomförts inom området mekaniska konstitutiva ekvationer. Inledningsvis behandlas begreppen fortifikation och konventionell vapenlast, varefter en kort sammanfattning av tillgängliga lösningsmetoder för penetrationsproblem följer. Fältteori för mekanik och termodynamik behandlas med kinematik för finita elastiska och inelastiska deformationer. Studien avslutas med en introduktion till mekaniska konstitutiva ekvationer med inriktning mot dynamiskt belastade solider.

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