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Abstract

We study approximate solutions of a slightly viscous conservation law in one dimension, constructed by two asymptotic expansions that are cut off after the third order terms. In the shock layer an inner solution is valid and an outer solution is valid elsewhere. The two solutions are matched in a matching region.

Based on the stability results in [10] we show that for a given time interval the difference between the approximate solutions and the true solution is not larger than $\mathcal{O}(\varepsilon)$, where ε is the viscosity coefficient.

Contents

1 Introduction

Computations in one and two dimensions presented in e.g. [1], [3] and [4] indicate that numerical solutions of conservation laws obtained by a higher order method degenerate in order of accuracy in space downstream of a shock layer.

Analysis of the source of the degeneracy have been made in e.g. [1], [2], [4] and [8] In [2] and [8] we study the steady state solution of slightly viscous hyperbolic systems of conservation laws with a lower order term. We base our results on the existence of matched asymptotic expansions. In the shock layer an inner solution is valid and an outer solution is valid elsewhere. The two solutions are matched together in the so called matching region. However, in [2] and [8] we do not prove that the asymptotic expansions exist.

In this report we consider

$$\begin{aligned} &u^{\varepsilon}_t + f(u^{\varepsilon})_x = \varepsilon u^{\varepsilon}_{xx}, \quad -\infty < x < \infty,\, t \geq 0, \\ &u^{\varepsilon}(x,0) = g^{\varepsilon}(x). \end{aligned} \tag{1}$$

where $u^{\varepsilon} \in \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}^n$, $x \in \mathbf{R}$ and ε is small. We shall investigate how well the first two terms of an asymptotic expansion approximates the solution of (1).

With $\varepsilon = \mathcal{O}(h)$, h being the grid size in a calculation, (1) is a so called model equation for a first order numerical scheme, see [13]. In this report we study an approximate solution of (1), v^{ε}, constructed by the first three terms of the inner and the outer expansions, respectively. The first term in the outer solution is the solution of the inviscid system. Based on the stability results in [10] we show that the difference between the approximate solution and the true solution is not larger than $\mathcal{O}(\varepsilon^{4\gamma-2})$, $\gamma \in (0.75, 1)$. This means that the difference between the solution of (1) and the corresponding inviscid system is to leading order the next term in the outer expansion. Since the result in [10] is for a traveling wave, corresponding to two constant states separated by a shock and moving with constant speed, we have to assume initial data such that the solution is close to the traveling wave.

In [14] the first terms in the expansions are used to analyze and eliminate the degeneracy to first order downstream of shocks observed in computations of time dependent solutions.

In [2], [8] and [14] the viscosity coefficient is a function of x. A larger viscosity coefficient is switched on in the shock region and a smaller is switched on elsewhere. In this report we only consider a constant viscosity problem. Preliminary studies indicate that the inclusion of a variable viscosity coefficient seems to effect the analysis presented in this report in a minor way. However, the stability analysis in [10] has to be extended.

The contents of this paper resembles the analysis presented in [6]. However, only weak shocks are considered in [6]. The results presented in this report holds for classical Lax shocks of arbitrary strength, under the assumption that a shock profile exists and is linearly stable.

2 Statement of the Problem

In this section we state the problem and the assumptions.

To begin with consider the inviscid problem

$$\begin{aligned} &u_t + f(u)_x = 0, \quad -\infty < x < \infty,\, t \geq 0, \\ &u(x,0) = g_0(x). \end{aligned} \tag{2}$$

Here, $g(x)$ is a piecewise smooth function. Let $f'(u)$ denote the Jacobian of the flux function. We assume that the eigenvalues of $f'(u)$, denoted $\lambda_i, i = 1, 2, ..., n$, are real, distinct and ordered in increasing order.

For the initial condition we make the following assumption.

Assumption 2.1 *The function g_0 is smooth except at $x = 0$ and it is constant outside a $\mathcal{O}(1)$-domain around the discontinuity, i.e. for some L*

$$g_0(x) = \begin{cases} u^+ & x > L \\ u^- & x < -L \end{cases}$$

We will consider the problem in some time interval, $0 \leq t \leq T$. We make the following assumption about the solution

Assumption 2.2 *The solution $u(x,t)$ of Eq (2) is a single shock solution up to time T. That is, u is smooth except at a point of discontinuity, $x = s(t)$, traveling with speed $\dot{s}$. There we require the solution to satisfy the Rankine–Hugoniot condition*

$$\dot{s}[u] = [f] \quad \text{at } x = s. \tag{3}$$

Here $[u] = u(s+0) - u(s-0)$, where $u(s \pm 0) = \lim_{\delta \to 0+} u(s \pm \delta)$. Also we assume that u^+, u^- together with some v_0 satisfies the Rankine-Hugoniot condition.

We call the discontinuity a $k-$shock if it in addition satisfies the Lax entropy condition [12],

$$\lambda_{k-1}^- < \dot{s} < \lambda_k^-,$$

$$\lambda_k^+ < \dot{s} < \lambda_{k+1}^+,$$

where $\lambda_k^\pm = \lambda_k(u(s \pm 0)))$. This means that exactly $n+1$ characteristics impinge on the shock. In this paper we only consider 1-shocks. Also, the matrix

$$D = \begin{pmatrix} S_{II}^+ & [u] \end{pmatrix} \tag{4}$$

is non-singular. Here S_{II}^+ are the eigenvectors of J^+ corresponding to the eigenvalues $\lambda_2^+, \lambda_3^+., ..., \lambda_n^+$.

Remark: For simplicity 1-shocks are considered in this paper. This is since in a 1-shock only one side of the shock are influenced by the first order error. For other k-shocks, however, both sides of the shock is polluted. The analysis is the same for all values of k, but the notation becomes more troublesome for $k \neq 1$.

The inviscid problem (2) is connected to Eq (1) in the following way. Many numerical solutions of (2) can be viewed as higher order accurate solutions of (1). See e. g. [13]. For strong shocks there are no general results on the existence of viscous profiles. Therefore we make the following assumption.

Assumption 2.3 *We assume that at each instant there is a viscous profile connecting the states on either side of the shock, that is for each $t \in [0, T]$ there exists a smooth solution of*

$$\begin{aligned} &-\dot{s}(t)\varphi_\xi + f(\varphi)_\xi = \varphi_{\xi\xi} \\ &\lim_{\xi\to\pm\infty} \varphi(\xi) = u(s(t)\pm, t). \end{aligned} \tag{5}$$

Also, the solution depends smoothly on boundary data. Especially we assume that a viscous profile corresponding to u^+, u^- and v_0 exists. Denote this profile by $\varphi_0(\xi)$.

The results of this paper are based on the stability result in [10] for φ_0. The following assumption is clearly necessary for the stability of the viscous profile φ_0.

Assumption 2.4 *Consider the eigenvalue problem*

$$\begin{aligned} \psi_{0\xi\xi} - \Big(A(\xi)\psi\Big)_\xi = \mu\psi_0, \quad \|\psi\|^2 =: \int_{-\infty}^{\infty} |\psi|^2 dx < \infty, \\ A(\xi) = f'(\varphi_0(\xi)) - v_0 I. \end{aligned} \tag{6}$$

We assume there are no eigenvalues with Re $\mu \geq 0$, $\mu \neq 0$, and the dimension of the eigenspace connected with the eigenvalue $\mu = 0$ is one.

Clearly, φ_0 will approach the end states exponentially fast and φ_0 and its derivatives will be uniformly bounded. Also, $\mu = 0$ is an eigenvalue with corresponding eigenfunction $\varphi_{0\xi}$.

Initial condition for (1) in the outer region is

$$g_0(x) + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \varepsilon^3 g_3(x).$$

Here g_0 is the inviscid initial condition and the functions g_l, $l = 1, 2, 3$ are smooth except at $x = 0$ and are non-zero only on an $\mathcal{O}(1)$-domain around the shock. That is,

$$g_i(x) = 0, \quad |x| \geq L,\ i = 1, 2, 3.$$

Let

$$\|g_i\|_{L_{2,1}} = \alpha_i, i = 0, 1, 2, 3,$$

where the point of discontinuity is excluded from the integral. In the inner region the initial conditions will be given by the construction. See Section 4.

3 Asymptotic Expansions

In this section we introduce the asymptotic expansions in the regions inside and outside the shock layer. Also, we derive the matching conditions valid in the matching regions. For a detailed presentation of matched asymptotic expansions we refer to [7] and [11].

Outside the shock region we assume that the solution can be expanded in powers of ε as

$$u^{\varepsilon}(x,t) \sim u_0(x,t) + \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + ... \tag{7}$$

The terms in the expansion (7) are solutions of the following equations

$$\mathcal{O}(1): \quad u_{0t} + f(u_0)_x = 0 \tag{8}$$
$$\mathcal{O}(\varepsilon): \quad u_{1t} + (f'(u_0)u_1)_x = u_{0xx} \tag{9}$$
$$\mathcal{O}(\varepsilon^2): \quad u_{2t} + (f'(u_0)u_2)_x = u_{1xx} - \frac{1}{2}(f''(u_0)(u_1,u_1))_x \tag{10}$$
$$\mathcal{O}(\varepsilon^3): \quad u_{3t} + (f'(u_0)u_3)_x = u_{2xx} - \frac{1}{2}(f''(u_0)(u_1,u_2))_x - \frac{1}{6}(f'''(u_0)(u_1,u_1,u_1))_x \tag{11}$$

Here $f''(u)(v,w)$ and $f'''(u)(v,v,v)$ are quadratic and cubic terms in the Taylor expansion of $f(u+v+w)$, respectively.

The initial data of (8) – (11) are

$$u_0(x,0) = g_0(x) \tag{12}$$
$$u_{1,2,3}(x,0) = g_{1,2,3}(x) \tag{13}$$

where $g_i \quad i = 0,1,2,3$ are introduced in the previous section.

By construction, for $0 \leq t \leq T$,

$$u_i(x,t) = 0, \quad x \notin [s(0) - L + T\lambda_1, s(0) + L + T\lambda_n], \quad i = 1,2,3.$$

Here $\lambda_i^{\pm}$ denotes the jth eigenvalue of $f'(u^{\pm})$, ordered in increasing order.

In the shock region the solution is expanded as

$$u^{\varepsilon}(x,t) \sim U_0(\xi,t) + \varepsilon U_1(\xi,t) + \varepsilon^2 U_2(\xi,t) + ... \tag{14}$$

Here ξ is the stretched variable

$$\xi = \frac{x - s(t)}{\varepsilon} + x_s(t,\varepsilon) \tag{15}$$

where

$$x_s(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + ... \tag{16}$$

is the expansion of the perturbation in the shock position. The equations for different orders in ε are

$$\begin{aligned}
\mathcal{O}(1): \quad & U_{0\xi\xi} + \dot{s}U_{0\xi} - f(U_0)_\xi &=& \; 0 && (17)\\
\mathcal{O}(\varepsilon): \quad & U_{1\xi\xi} + \dot{s}U_{1\xi} - (f'(U_0)U_1)_\xi &=& \; \dot{x}_0 U_{0\xi} + U_{0t} && (18)\\
\mathcal{O}(\varepsilon^2): \quad & U_{2\xi\xi} + \dot{s}U_{2\xi} - (f'(U_0)U_2)_\xi &=& \; \dot{x}_1 U_{0\xi} + \dot{x_0} U_{1\xi} + U_{1t} + \\
& && \; \frac{1}{2}(f''(U_0)(U_1,U_1))_\xi && (19)\\
\mathcal{O}(\varepsilon^3): \quad & U_{3\xi\xi} + \dot{s}U_{3\xi} - (f'(U_0)U_3)_\xi &=& \; \dot{x}_2 U_{0\xi} + \dot{x_1} U_{1\xi} + x_0 U_{2t} + U_{2t} + \\
& && \; \frac{1}{2}(f''(U_0)(U_1,U_2))_\xi + \\
& && \; \frac{1}{6}(f'''(U_0)(U_1,U_1,U_1))_\xi && (20)
\end{aligned}$$

Both expansions are valid in the matching regions and are connected by the so called matching conditions. As $\xi \to \pm\infty$ the inner solution shall approach values of the outer solution at $x = s \pm 0$. The outer solution expressed in the variable ξ is

$$\begin{aligned}
u^\varepsilon \;\sim\; & u_0(\varepsilon\xi + s(t) - \varepsilon x_s, t) + \varepsilon u_1(\varepsilon\xi + s(t) - \varepsilon x_s, t) + \varepsilon^2 u_2(\varepsilon\xi + s(t) - \varepsilon x_s, t) + \\
& \varepsilon^3 u_3(\varepsilon\xi + s(t) - \varepsilon x_s, t) + \ldots
\end{aligned}$$

Taylor expansion around $x = s \pm 0$ yields

$$\begin{aligned}
u^\varepsilon(x,t) \;\sim\; & u_0(s \pm 0, t) + \varepsilon(\xi - x_s)u_{0x}(s \pm 0, t) + \\
& \frac{1}{2}\varepsilon^2(\xi - x_s)^2 u_{0xx} + \frac{1}{6}\varepsilon^3(\xi - x_s)^3 u_{0xxx} + \\
& \varepsilon(u_1(s \pm 0, t) + \varepsilon(\xi - x_s)u_{1x} + \frac{1}{2}\varepsilon^2(\xi - x_s)^2 u_{1xx}) + \\
& \varepsilon^2(u_2(s \pm 0, t) + \varepsilon(\xi - x_s)u_{2x}) + \varepsilon^3 u_3(s \pm 0, t) + \mathcal{O}(\varepsilon^4)
\end{aligned}$$

It follows that the matching conditions are,

$$\begin{aligned}
U_0(\xi,t) &= u_0(s \pm 0, t) + o(1) && (21)\\
U_1(\xi,t) &= u_1(s \pm 0, t) + (\xi - x_0)u_{0x}(s \pm 0, t) + o(1) && (22)\\
U_2(\xi,t) &= u_2(s \pm 0, t) + (\xi - x_0)u_{1x}(s \pm 0, t) + \frac{1}{2}(\xi - x_0)^2 u_{0xx} - \\
& \quad x_1 u_{0x} + o(1) && (23)\\
U_3(\xi,t) &= u_3(s \pm 0, t) + (\xi - x_0)u_{2x}(s \pm 0, t) - x_1 u_{1x}(s \pm 0, t) - \\
& \quad x_2 u_{0x}(s \pm 0, t) - (\xi - x_0)x_1 u_{0xx} + \frac{1}{2}(\xi - x_0)^2 u_{1xx} + \\
& \quad \frac{1}{6}(\xi - x_0)^3 u_{0xxx} + o(1) && (24)
\end{aligned}$$

in the matching region. Note that the $o(1)$-terms are exponentially small, i.e. $e^{-|\xi|}$.

We also need boundary conditions for the outer solution at the shock. The boundary conditions for u_0 are the Rankine-Hugoniot condition (3) with $u = u_0$.

No boundary conditions are needed for the upstream branch of u_1 at $x = s - 0$, since the flow is supersonic upstream and all characteristics go into the shock.

The boundary conditions at $x = s + 0$ are achieved in the following way. Let x_m^- and x_m^+ be points in the matching regions upstream and downstream of the shock, respectively. By integrating (1) over the interval $[x_m^-, x_m^+]$ we have

$$\int_{x_m^-}^{x_m^+} u_t^\varepsilon + [f(u^\varepsilon)]_{x_m^-}^{x_m^+} - \varepsilon[u_x^\varepsilon]_{x_m^-}^{x_m^+} = 0 \tag{25}$$

Since x_m^- and x_m^+ are functions of t it follows that

$$\int_{x_m^-}^{x_m^+} u_t^\varepsilon dx = \frac{d}{dt}\int_{x_m^-}^{x_m^+} u^\varepsilon dx - \frac{dx_m^+}{dt}u^\varepsilon(x_m^+) + \frac{dx_m^-}{dt}u^\varepsilon(x_m^-) \tag{26}$$

The matching points moves with the speed of the viscous shock layer, that is

$$\frac{dx_m^\pm}{dt} = \frac{dx_m^\pm}{dt} = \dot{s} - \varepsilon\dot{x}_0 - \varepsilon^2\dot{x}_1 + \mathcal{O}(\varepsilon^3) \tag{27}$$

With the change of variable

$$\xi = \frac{x - s(t)}{\varepsilon} + x_s(t)$$

and the use of the inner expansion it follows that

$$\int_{x_m^-}^{x_m^+} u_t^\varepsilon dx = \varepsilon\frac{d}{dt}\int_{\tilde{x}_m^-}^{\tilde{x}_m^+} (U_0 + \varepsilon U_1 + \varepsilon^2 U2 + \mathcal{O}(\varepsilon^3))d\xi - \\ (\dot{s} - \varepsilon\dot{x}_0 - \varepsilon^2\dot{x}_1 + \mathcal{O}(\varepsilon^3))[u^\varepsilon]_{x_m^-}^{x_m^+}$$

By using the matching conditions we obtain

$$\int_{x_m^-}^{x_m^+} u_t^\varepsilon dx = \varepsilon\frac{d}{dt}\int_{\tilde{x}_m^-}^{\tilde{x}_m^+} (U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \mathcal{O}(\varepsilon^3))d\xi - \\ (\dot{s} - \varepsilon\dot{x}_0 - \varepsilon^2\dot{x}_1 + \mathcal{O}(\varepsilon^3))([u_0] + \varepsilon[u_1 + (\xi - x_0)u_{0x}] + \mathcal{O}(\varepsilon^2))$$

Here $[\cdot] := [\cdot]_{x=s-0}^{s+0}$.

Also, by using the inner expansion and the matching conditions, the second term in Eq (25) can be written as

$$\begin{aligned}[f(u^\varepsilon)]_{x_m^-}^{x_m^+} &= f(u_0^+ + \varepsilon(u_1(s+0,t) + (\xi - x_0)u_{0x}(s+0,t)) + \mathcal{O}(\varepsilon^2)) - \\ &\quad f(u_0^- + \varepsilon(u_1(s-0,t) + (\xi - x_0)u_{0x}(s-0,t)) + \mathcal{O}(\varepsilon^2)) \\ &= [f(u_0)] + \varepsilon[f'(u_0)(u_1 + (\xi - x_0)u_{0x})] + \mathcal{O}(\varepsilon^2)\end{aligned}$$

Similarly, the last term in Eq (25) can be written as

$$\varepsilon[u_x^\varepsilon]_{x_m^-}^{x_m^+} = \varepsilon[u_{0x}] + \varepsilon^2[u_1x + (\xi - x_0)u_{0xx}] + \mathcal{O}(\varepsilon^3) \tag{28}$$

It follows from Eq (25) that for the $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$-terms, respectively, it should hold that

$$\mathcal{O}(1): \qquad -\dot{s}[u_0] + [f(u_0)] = 0 \tag{29}$$

$$\mathcal{O}(\varepsilon): \int_{-\infty}^{\infty} U_1 d\xi - \dot{s}[u_1 + (\xi - x_0)u_{0x}] + \dot{x_0}[u_0] + \tag{30}$$

$$[f'(u_0)(u_1 + \xi - x_0)u_{0x}] - [u_{0x}] + \mathbf{o}(1) = 0 \tag{31}$$

In (31) we have used that U_1 converges to its boundary states exponentially fast. Hence, $\tilde{x}_m^{\pm}$ can be replaced with $\pm\infty$ with only introducing exponentially small errors.

Since u_0 satisfy the Rankine–Hugoniot condition by assumption 2.2 Eq (29) is fulfilled.

After some elementary calculus and algebra we end up with the boundary condition for u_1 at $x = s + 0$ from Eq (31), namely

$$u_1(s+0) = -(J_+ - \dot{s}I)^{-1}(I_3 + \frac{d}{dt}(x_0[u_0]) + [u_{0x}] + (\dot{s}I - J_-)u_1(s-0))$$

where

$$I_3 = \frac{d}{dt}(\int_{-\infty}^{0} (U_0 - u^+)d\xi + \int_{0}^{\infty} (U_0 - u^-)d\xi)$$

and

$$J_+ = f'(u_0(s+0)) \quad J_- = f'(u_0(s-0))$$

Finally, the equations for $u_1(s+0)$ and $x_0(t)$ are

$$\begin{aligned} (J_+ - \dot{s}I)u_1(s+0) + (x_0[u_0])_t &= -(I_3 + [u_{0x}] + && (32)\\ &\quad (\dot{s}I - J_-)u_1(s-0)) && (33)\\ x_1(0) &= 0 \end{aligned}$$

or equivalently

$$\begin{pmatrix} w_{II}^+ \\ \dot{x}_1 \end{pmatrix} = \begin{pmatrix} \Lambda_{II}^+ - \dot{s}I & 0 \\ 0 & -1 \end{pmatrix}^{-1} D^{-1}H(x,t) \tag{34}$$

$$x_1(0) = 0 \tag{35}$$

where

$$H(x,t) = (-I_3 + x_0[u_0]_t + [u_{0x}] + (\dot{s}I - J_-)u_1(s-0)) - S_I^+(\lambda_1^+ - \dot{s})w_I^+)$$

In (34) w_{II}^+ are the characteristic variables of $u_1(s+0)$ going out of the shocks, w_I^+ is the characteristic variable going into the shock, $\Lambda_{II}^+ = \text{diag}(\lambda_2^+, ..., \lambda_n^+)$, S_I^+ is the eigenvector of J^+ corresponding to the eigenvalue λ_1^+ and D is defined by (4). That is, in general $u_1(x,t) \neq 0$ downstream of the shock.

By the assumptions on u_0 and U_0 it follows that the forcing in (34) is a smooth function of t. Thus, one can show by standard methods, see [9], that if g_1 is smooth away from $x = 0$, then u_1 and its derivatives are smooth except at $x = s(t)$. Condition (34) is the solvability condition for Eq (18). Thus (18) has a smooth solution which approaches its limiting shape exponentially fast. A proof of this can be found in [10]. Since the forcing in (18) depends smoothly on t so will U_1.

The procedure can be continued. Boundary conditions for u_p, $p \geq 2$ at $x = x^+$ are derived analogously by including higher order terms in ε in the above derivation. These boundary conditions are the solvability conditions for the equations $U_p, p \geq 2$. Smoothness follows as before.

Note that for the special case

$$g_0(x) = \begin{cases} u^+ & x > 0 \\ u^- & x < 0 \end{cases}$$

the forcings in (34) and in (9), respectively, vanish since $u_{0x} = u_{0xx} = 0$ and $I_3 = 0$. By standard energy estimates there is a constant $K(T)$ such that for $t \in [0, T]$

$$||u_1||^2_{L_{2,1}[-\infty,s(t)]} + ||u_1||^2_{L_{2,1}[s(t),\infty]} \leq K(T)(||g_1||^2_{L_{2,1}[-\infty,s(t)]} + ||g_1||^2_{L_{2,1}[s(t),\infty]}) \tag{36}$$

4 The Approximate Solution

In this section we construct an approximate solution to Eq (1) by matching truncated inner and outer solutions presented in the previous section.

We define

$$\begin{aligned} I(x,t) &= U_0(\frac{x-s(t)}{\varepsilon}+x_s(t),t)+\varepsilon U_1(\frac{x-s(t)}{\varepsilon}+x_s,t)+ \\ &\quad \varepsilon^2 U_2(\frac{x-s(t)}{\varepsilon}+x_s,t)+\varepsilon^3 U_3(\frac{x-s(t)}{\varepsilon}+x_s,t) \\ x_s(t) &= x_0+\varepsilon x_1(t)+\varepsilon^2 x_2(t) \\ O(x,t) &= u_0(x,t)+\varepsilon u_1(x,t)+\varepsilon^2 u_2(x,t)+\varepsilon^3 u_3(x,t) \end{aligned}$$

where $u_i,\ i=0,1,2,3$ satisfies (8)-(11), and $U_i,\ i=0,1,2,3$ satisfies (17)-(20), respectively. Also, the boundary conditions at $x=x_m^+$ and the matching conditions are fulfilled.

We introduce the approximate solution to (1), denoted v^ε, by

$$v^\varepsilon(x,t)=m(\frac{x-s(t)}{\varepsilon^\gamma})I(x,t)+(1-m(\frac{x-s(t)}{\varepsilon^\gamma}))O(x,t)+d(x,t) \qquad (37)$$

where $m(y)\in C_0^\infty(\mathbf{R})$, $0\le m(y)\le 1$

$$m(y)=\begin{cases} 1 & |y|\le 1 \\ 0 & |y|\ge 2. \end{cases}$$

Hence, γ is a parameter that determines the rate of the switch between the inner and outer solution, that is the width of the matching region. From the matching conditions, (21) – (24) it follows that $\gamma\in(\frac{3}{4},1)$, see [5]. The term $d(x,t)$ contains higher order corrections which will be determined below.

The approximate solution, v^ε, satisfies

$$\begin{aligned} &v_t^\varepsilon+f(v^\varepsilon)_x=\varepsilon v_{xx}^\varepsilon+\textstyle\sum_{i=1}^4 q_i \\ &v^\varepsilon(x,0)=u^\varepsilon(x,0)=g(x). \end{aligned}$$

where

$$\begin{aligned} q_1 &= (1-m)\{(f(O(x,t))-f(u_0)-\varepsilon f'(u_0)u_1-\varepsilon^2 f'(u_0)u_2-\varepsilon^3 f'(u_0)u_3 \\ &\quad -\varepsilon^2\tfrac{1}{2}f''(u_0)(u_1,u_1)- \\ &\quad \varepsilon^3\tfrac{1}{2}f''(u_0)(u_1,u_2)-\varepsilon^3\tfrac{1}{6}f'''(u_0)(u_1,u_1,u_1))_x-\varepsilon^4 u_{3xx}\} \\ q_2 &= m\{(f(I(x,t))-f(U_0)-\varepsilon f'(U_0)U_1-\varepsilon^2 f'(U_0)U_2- \\ &\quad \varepsilon^3 f'(U_0)U_3-\varepsilon^2\tfrac{1}{2}f''(U_0)(U_1,U_1) \\ &\quad -\varepsilon^3\tfrac{1}{2}f''(U_0)(U_1,U_2)-\varepsilon^3\tfrac{1}{6}f'''(U_0)(U_1,U_1,U_1))_x+\varepsilon^3 U3,t+ \\ &\quad \varepsilon^4(\dot{x_2}U_1+\dot{x_1}U_2+\varepsilon\dot{x_2}U_2+\dot{x_0}U_3+\varepsilon\dot{x_1}U_3+\varepsilon^2\dot{x_2}U_3)_x\} \\ q_3 &= m_t(I(x,t)-O(x,t))-\varepsilon m_{xx}(I(x,t)-O(x,t)) \\ &\quad 2\varepsilon m_x(I(x,t)-O(x,t))_x+m_x(f(I(x,t))-f(O(x,t)))+ \\ &\quad f(mI(x,t)+(1-m)O(x,t))-(mf(I(x,t))+(1-m)f(O(x,t))))_x \\ q_4 &= d_t-\varepsilon d_{xx}(f(v^\varepsilon)-f(v^\varepsilon-d))_x \end{aligned}$$

Here $f''(u)(v,w)$ and $f'''(u)(v,v,v)$ are quadratic and cubic terms in the Taylor expansion of $f(u+v+w)$, respectively.

We have that

$$\begin{array}{l} supp\ q_1 \subseteq \{(x,t) : \varepsilon^\gamma \le |x-s| \le \mathcal{O}(1), 0 \le t \le T\} \\ \frac{\partial^l}{\partial x^l} q_1(x,t) = \mathcal{O}(1)\varepsilon^{4-l\gamma} \quad l=0,1,2,3 \end{array} \tag{38}$$

Also,

$$\begin{array}{l} supp\ q_2 \subseteq \{(x,t) : |x-s| \le 2\varepsilon^\gamma, 0 \le t \le T\} \\ \frac{\partial^l}{\partial x^l} q_2(x,t) = \mathcal{O}(1)\varepsilon^{3-l\gamma} \quad l=0,1,2,3 \end{array} \tag{39}$$

and

$$\begin{array}{l} supp\ q_3 \subseteq \{(x,t) : \varepsilon^\gamma \le |x-s| \le 2\varepsilon^\gamma, 0 \le t \le T\} \\ \frac{\partial^l}{\partial x^l} q_3(x,t) = \mathcal{O}(1)\varepsilon^{(3-l)\gamma} \quad l=0,1,2,3 \end{array} \tag{40}$$

In (40) we have used the estimate

$$\partial_x^l(I(x,t)-O(x,t)) = \mathcal{O}(1)\varepsilon^{(4-l)\gamma} \quad \text{on } \{(x,t) : \varepsilon^\gamma \le |x-s(t)| \le 2\varepsilon^\gamma, t \in [0,T]\}. \tag{41}$$

which can be obtained from the matching conditions.

Let $d(x,t)$ be the solution of

$$\begin{array}{l} d_t = \varepsilon d_{xx} - \sum_{i=1}^3 q_i(x,t) \\ d(x,0) = 0. \end{array} \tag{42}$$

or in the scaled variables $\tilde{x} = \frac{x-s(t)}{\varepsilon}$ $\quad \tilde{t} = t/\varepsilon$.

$$\begin{array}{l} d_{\tilde{t}} = \dot{s}(\varepsilon\tilde{t})d_{\tilde{x}} + d_{\tilde{x}\tilde{x}} - \varepsilon\sum_{i=1}^3 q_i(\tilde{x},\tilde{t}) \\ d \to 0 \quad \text{as } \tilde{x} \to \pm\infty \\ d(\tilde{x},0) = 0. \end{array} \tag{43}$$

Remark The initial data $d(x,0)$ is allowed to be of $\mathcal{O}(\varepsilon^4)$ in the shock region and zero elsewhere.

The equation for v^ε hence becomes

$$v_t^\varepsilon + f(v^\varepsilon)_x = \varepsilon v_{xx}^\varepsilon + (f(v^\varepsilon) - f(v^\varepsilon - d))_x.$$

Below, we use the notation

$$\begin{array}{ll} ||d(\cdot,t)||_{L_2,p}^2 = \sum_{l=0}^p \int_{-\infty}^{\infty} |\partial_x^l d(x,t)|^2 dx & ||d(\cdot,t)||_{\tilde{L}_2,p}^2 = \sum_{l=0}^p \int_{-\infty}^{\infty} |\partial_{\tilde{x}}^l d(\tilde{x},t)|^2 d\tilde{x} \\ ||d(\cdot,t)||_{L_1,p} = \sum_{l=0}^p \int_{-\infty}^{\infty} |\partial_x^l d(x,t)| dx & ||d(\cdot,t)||_{\tilde{L}_1,p} = \sum_{l=0}^p \int_{-\infty}^{\infty} |\partial_{\tilde{x}}^l d(\tilde{x},t)| d\tilde{x} \\ ||d(\cdot,t)||_\infty = \sup_x |d(\cdot,t)| & \end{array}$$

If $p=0$ we suppress it.

For future reference we here present some estimates on $d(\tilde{x},\tilde{t})$.

Lemma 4.1 *Let $d(\tilde{x},\tilde{t})$ be the solution of Eq (43). The following estimates hold for $\tilde{t} \in [0,T/\varepsilon]$.*

$$||d||_{\tilde{L}_2,2} \le \mathcal{O}(1)\varepsilon^{(7\gamma-1)/2} \tag{44}$$

$$||d||_{\tilde{L}_1,2} \le \mathcal{O}(1)\varepsilon^{4\gamma-1} \tag{45}$$

$$||d||_\infty \le \mathcal{O}(1)\varepsilon^{4\gamma} \tag{46}$$

Proof Firstly, by an energy estimate derived from (42) it holds that

$$||d(\cdot,t)||_{L_2} \leq \int_0^t ||q(\cdot,\tau)||_{L_2} d\tau$$

where $q = q_1 + q_2 + q_3$. By definition

$$||d(\cdot,t)||_{L_2} = (\int_{-\infty}^{\infty} |d(x,t)|^2 dx)^{1/2} = \sqrt{\varepsilon}(\int_{-\infty}^{\infty} |d(\tilde{x},t)|^2 d\tilde{x})^{1/2} = \sqrt{\varepsilon}||d(\cdot,\tilde{t})||_{\tilde{L}_2}$$

Hence

$$||d(\cdot,\tilde{t})||_{\tilde{L}_2} = \frac{1}{\sqrt{\varepsilon}}||d(\cdot,t)||_{L_2} \leq \frac{1}{\sqrt{\varepsilon}} \int_0^t ||q(\cdot,\tau)||_{L_2} d\tau$$

Now, by (38), (39) and (40), respectively,

$$\begin{aligned}
||q_1(\cdot,\tau)||_{L_2} &= (\int_{\varepsilon^\gamma < |x-s| \leq \mathcal{O}(1)} |q_1(x,\tau)|^2 dx)^{1/2} \leq \mathcal{O}(1)\varepsilon^4 \\
||q_2(\cdot,\tau)||_{L_2} &= (\int_{|x-s| \leq 2\varepsilon^\gamma} |q_2(x,\tau)|^2 dx)^{1/2} \leq \mathcal{O}(1)\varepsilon^{3+\gamma/2} \\
||q_3(\cdot,\tau)||_{L_2} &= (\int_{\varepsilon^\gamma \leq |x-s| \leq 2\varepsilon^\gamma} |q_3(x,\tau)|^2 dx)^{1/2} \leq \mathcal{O}(1)\varepsilon^{7\gamma/2}
\end{aligned}$$

Since $t = \mathcal{O}(1)$ it follows that

$$||d(\cdot,\tilde{t})||_{\tilde{L}_2} \leq \mathcal{O}(1)\varepsilon^{(7\gamma-1)/2} \tag{47}$$

To estimate $d_{\tilde{x}}$ and $d_{\tilde{x}\tilde{x}}$ we first estimate d_x and d_{xx}. Differentiation of Eq (42) w.r.t. x and partial integration it follows that

$$\frac{d}{dt}||\partial_x^l d||_{L_2}^2 \leq -\varepsilon||\partial_x^{l+1} d||_{L_2}^2 + ||\partial_x^{l+1} d||_{L_2}^2 ||\partial^{l-1} q||_{L_2} \leq \frac{1}{\varepsilon}||\partial^{l-1} q||_{L_2}^2 \quad l = 1,2$$

Hence

$$||\partial_x^l d||_{L_2} \leq \frac{1}{\sqrt{\varepsilon}}(\int_0^t ||\partial^{l-1} q(\cdot,\tau)||_{L_2}^2 d\tau)^{1/2}$$

Now,

$$\begin{aligned}
||\partial_x d(\cdot,t)||_{L_2} &= (\int_{-\infty}^{\infty} |d_x(x,t)|^2 dx)^{1/2} = (\varepsilon \int_{-\infty}^{\infty} \tfrac{1}{\varepsilon^2}|d_{\tilde{x}}(\tilde{x},\tilde{t})|^2 d\tilde{x})^{1/2} = \\
&\quad \tfrac{1}{\sqrt{\varepsilon}}||\partial_{\tilde{x}} d(\cdot,\tilde{t})||_{\tilde{L}_2} \\
||\partial_x^2 d(\cdot,t)||_{L_2} &= \tfrac{1}{\varepsilon^{3/2}}||\partial_{\tilde{x}}^2 d(\cdot,\tilde{t})||_{\tilde{L}_2}
\end{aligned}$$

It follows that

$$||\partial_{\tilde{x}}^l d||_{\tilde{L}_2} = \varepsilon^{l-1}\sqrt{\varepsilon}||\partial_x^l d||_{L_2} \leq \varepsilon^{l-1}(\int_0^t ||\partial^{l-1} q(\cdot,\tau)||_{L_2}^2 d\tau)^{1/2}$$

By (38),(39)and (40), respectively,

$$\begin{aligned} ||\partial_x^l q_1(\cdot,t)||_{L_2} &= \mathcal{O}(\varepsilon^{4-l\gamma}) \\ ||\partial_x^l q_2(\cdot,t)||_{L_2} &= \mathcal{O}(\varepsilon^{3-(l-1/2)\gamma}) \\ ||\partial_x^l q_3(\cdot,t)||_{L_2} &= \mathcal{O}(\varepsilon^{(7/2-l)\gamma}) \end{aligned}$$

and since $t = \mathcal{O}(1)$, it holds that

$$||\partial_{\tilde{x}}^l d||_{\tilde{L}_2} \leq \mathcal{O}(\varepsilon^{(l-1)(1-\gamma)+7\gamma/2}) \quad \text{l=1,2} \tag{48}$$

Hence

$$||d(\cdot,\tilde{t})||_{\tilde{L}_2,2} \leq \mathcal{O}(\varepsilon^{(7\gamma-1)/2}) \tag{49}$$

We now proceed to estimate $||d(\cdot,\tilde{t})||_{\tilde{L}_1}$. We let $g(x,t) = (4\varepsilon\pi t)^{1/2} exp\{-x^2/(4\varepsilon t)\}$ and $G(x,t) = diag(g(x,t), g(x,t), ..., g(x,t))$. The solution of (42) is hence

$$d(x,t) \;=\; \int_0^t \int_{-\infty}^{\infty} G(x-y,t-\tau)(-\sum_{j=0}^{3} q_j(y,\tau))dyd\tau.$$

It follows that

$$||d(\cdot,t)||_{L_1} = \int_{-\infty}^{\infty} |\int_0^t \int_{-\infty}^{\infty} G(x-y,t-\tau)q(y,\tau)dyd\tau|dx \leq$$

$$\int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x-y,t-\tau)dx|q(y,\tau)|dyd\tau = \int_0^t ||q(\cdot,t)||_{L_1} d\tau$$

where as before $q(x,t) = q_1(x,t) + q_2(x,t) + q_3(x,t)$. By definition

$$||d(\cdot,t)||_{L_1} = \int_{-\infty}^{\infty} |d(x,t)|dx = \varepsilon \int_{-\infty}^{\infty} |d(\tilde{x},\tilde{t})|d\tilde{x} = \varepsilon ||d(\cdot,\tilde{t})||_{\tilde{L}_1}$$

Hence

$$||d(\cdot,\tilde{t})||_{\tilde{L}_1} = \frac{1}{\varepsilon}||d(\cdot,t)||_{L_1} \leq \frac{1}{\varepsilon}\int_0^t ||q(\cdot,\tau)||_{L_1} d\tau$$

Since

$$\begin{aligned} ||q_1||_{L_1} &= \mathcal{O}(1)\varepsilon^4 \\ ||q_2||_{L_1} &= \mathcal{O}(1)\varepsilon^{3+\gamma} \\ ||q_3||_{L_1} &= \mathcal{O}(1)\varepsilon^{4\gamma} \end{aligned}$$

it follows that

$$||d(\cdot,\tilde{t})||_{\tilde{L}_1} \leq \mathcal{O}(1)\varepsilon^{4\gamma-1} \tag{50}$$

Differentiation w.r.t. x of Eq (42) yields

$$(\partial_x^l d)_t = \varepsilon(\partial_x^l d)_{xx} + \partial_x^l q \quad l = 1,2$$

By partial integration it follows that

$$||\partial_x^l d||_{L_1} = \int_{-\infty}^{\infty} |\int_0^t \int_{-\infty}^{\infty} \frac{x-y}{2\varepsilon(t-\tau)} G(x-y,t-\tau)\partial_x^{l-1} q(y,\tau)dyd\tau|dx \leq$$

$$\int_0^t \frac{1}{\sqrt{4\pi\varepsilon(t-\tau)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_x^{l-1} q(y,\tau)|dyd\tau = \mathcal{O}(\varepsilon^{-1/2}) \int_0^t ||\partial_x^{l-1} q(\cdot,t)||_{L_1} d\tau$$

By definition

$$\begin{aligned} ||\partial_x d||_{L_1} &= \int_{-\infty}^{\infty} |\partial_x d|dx = \int_{-\infty}^{\infty} |\partial_{\tilde{x}} d|d\tilde{x} = ||\partial_{\tilde{x}} d||_{\tilde{L}_1} \\ ||\partial_x^2 d||_{L_1} &= \int_{-\infty}^{\infty} |\partial_x^2 d|dx = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} |\partial_{\tilde{x}}^2 d|d\tilde{x} = \frac{1}{\varepsilon}||\partial_{\tilde{x}}^2 d||_{\tilde{L}_1} \end{aligned}$$

hence

$$||\partial_{\tilde{x}}^l d||_{\tilde{L}_1} = \varepsilon^{(l-1)}||\partial_x^l d||_{L_1}$$

With

$$\begin{aligned} ||\partial_x q_1||_{L_1} &= \mathcal{O}(1)\varepsilon^{4-\gamma} \\ ||\partial_x q_2||_{L_1} &= \mathcal{O}(1)\varepsilon^{3} \\ ||\partial_x q_3||_{L_1} &= \mathcal{O}(1)\varepsilon^{3\gamma} \end{aligned}$$

it follows that

$$||\partial_{\tilde{x}}^l d(\cdot,\tilde{t})||_{\tilde{L}_1} \leq \mathcal{O}(1)\varepsilon^{(l-1)(1+\gamma)-1/2+4\gamma} \quad l = 1,2 \tag{51}$$

and hence

$$||d(\cdot,\tilde{t})||_{\tilde{L}_{1,2}} \leq \mathcal{O}(1)\varepsilon^{4\gamma-1} \tag{52}$$

Finally, we see that

$$||d(\cdot,t)||_\infty = \sup_x |\int_0^t \int_{-\infty}^{\infty} G(x-y,t-\tau)q(y,\tau)dyd\tau| \leq$$

$$\sup_x \int_0^t \sup_y |q(y,\tau)| \int_{-\infty}^{\infty} G(x-y,t-\tau)dyd\tau = \int_0^t ||q(\cdot,t)||_\infty d\tau$$

By (38)–(40) we have that

$$||d(\cdot,t)||_\infty \leq \mathcal{O}(1)\varepsilon^{4\gamma}$$

which proves the lemma. ■

5 Stability Analysis

In this section we shall show that in a given time interval $0 \leq t \leq T$ the difference between the approximate solution, v^ε, that was defined in (37), and the solution u^ε of (1), is small.

We define $w(x,t)$ as

$$w(x,t) = u^\varepsilon(x,t) - v^\varepsilon(x,t)$$

It follows that w satisfies the equation

$$\begin{aligned} &w_t + (f'(v^\varepsilon)w)_x + Q(v^\varepsilon, w)_x = \varepsilon w_{xx} + (f(v^\varepsilon - d) - f(v^\varepsilon))_x \\ &w(x,0) = 0 \end{aligned} \tag{53}$$

Here $Q(v^\varepsilon, w) = f(u^\varepsilon) - f(v^\varepsilon) - f'(v^\varepsilon)w$, which by the smoothness assumptions on f satisfies $|Q| \leq K|w|^2$ and $|Q_x| \leq K|w||w_x|$ for small w.

Below, we will use the stability results presented in [10]. However, to apply these results, the Jacobian in the second term in (53) should be evaluated along a traveling wave solution of (1). We will use φ_0, the profile connecting u^+ with u^- and moving with speed v_0.

Introducing φ_0 and the scaled variables $\tilde{x} = \frac{x - v_0 t}{\varepsilon}, \tilde{t} = \frac{t}{\varepsilon}$ into (53) yields

$$w_{\tilde{t}} + (A(\tilde{x})w)_{\tilde{x}} + Q(v^\varepsilon, w)_{\tilde{x}} + (B(\tilde{x}, \tilde{t})w)_{\tilde{x}} = w_{\tilde{x}\tilde{x}} + F(\tilde{x}, \tilde{t})_{\tilde{x}}. \tag{54}$$

Here $A(\tilde{x}) = f'(\varphi_0(\tilde{x}) - v_0 I$, $B(\tilde{x}, \tilde{t}) = f'(v^\varepsilon(\varepsilon(\tilde{x} + v_0\tilde{t}), \varepsilon\tilde{t})) - f'(\varphi_0(\tilde{x}))$ and $F = f(v^\varepsilon - d) - f(v^\varepsilon)$. By the smoothness of f and the properties of φ_0 and v^ε we have

$$\begin{aligned} &|B| \leq C|v^\varepsilon - \varphi_0|, \\ &|B_{\tilde{x}}| \leq C(|v^\varepsilon - \varphi_0| + |v^\varepsilon_{\tilde{x}} - \varphi_{\tilde{x}}|), \\ &|F| \leq C|d|, \\ &|F_{\tilde{x}}| \leq C(|d| + |d_{\tilde{x}}|). \end{aligned} \tag{55}$$

From [10] we have the following Lemma

Lemma 5.1 *Consider (54) with $Q \equiv 0$ and $B \equiv 0$. Under the assumptions 2.2, 2.3, 2.4 there is a constant R, independent of $\tilde{T} = T/\varepsilon$ and F such that the solution satisfies the estimate*

$$\begin{aligned} &\int_0^{\tilde{T}} \|u(\cdot,\tilde{t})\|^2_{2,2} + \|u_t(\cdot,\tilde{t})\|^2_{2,1} d\tilde{t} \\ &\leq R\left(\big(\int_0^{\tilde{T}} \|F(\cdot,\tilde{t})\|_{1,1} d\tilde{t}\big)^2 + \int_0^{\tilde{T}} \|F(\cdot,\tilde{t})\|^2_{2,1} d\tilde{t} \right). \end{aligned} \tag{56}$$

We need to estimate the right hand side of (56). By (55) and Lemma 4.1

$$\int_0^{T/\varepsilon}\int_{-\infty}^{\infty} |F(\tilde{x},\tilde{t})|^2 d\tilde{x}d\tilde{t} \le C\int_0^{T/\varepsilon} \|d(\cdot,\tilde{t}\|^2_{\tilde{L}_2} d\tilde{t} \le C_1\varepsilon^{7\gamma-2},$$

$$\int_0^{T/\varepsilon}\int_{-\infty}^{\infty} |F_{\tilde{x}}(\tilde{x},\tilde{t})|^2 d\tilde{x}d\tilde{t} \le C\int_0^{T/\varepsilon} \|d(\cdot,\tilde{t})\|^2_{\tilde{L}_{2,1}} d\tilde{x} \le c_2\varepsilon^{7\gamma-2},$$

$$\int_0^{T/\varepsilon}\int_{-\infty}^{\infty} |F(\tilde{x},\tilde{t})| d\tilde{x}d\tilde{t} \le C\int_0^{T/\varepsilon} \|d(\cdot,\tilde{t})\|_{\tilde{L}_1} d\tilde{t} \le c_3\varepsilon^{4\gamma-2},$$

$$\int_0^{T/\varepsilon}\int_{-\infty}^{\infty} |F_{\tilde{x}}(\tilde{x},\tilde{t})| d\tilde{x}d\tilde{t} \le C\int_0^{T/\varepsilon} \|d(\cdot,\tilde{t})\|_{L_{1,1}} d\tilde{t} \le C_4\varepsilon^{4\gamma-2} \quad (57)$$

It follows that

$$\int_0^{\tilde{T}} \|F(\cdot,\tilde{t})\|^2_{\tilde{L}_{2,1}} d\tilde{t} + \big(\int_0^{\tilde{T}} \|F(\cdot,\tilde{t})\|_{\tilde{L}_{1,1}} d\tilde{t}\big)^2 \le K(\varepsilon^{7\gamma-2}+\varepsilon^{8\gamma-4}) \le 2K\varepsilon^{8\gamma-4}. \quad (58)$$

The last inequality follows since $\gamma \in (0.75, 1)$.

We expect the nonlinear problem to satisfy a similar estimate. Therefore we introduce the scaling

$$w = \delta\tilde{w}, \quad F = \delta\tilde{F}, \quad \delta = \varepsilon^{4\gamma-2}, \quad (59)$$

in (54), yielding

$$\tilde{w}_{\tilde{t}} + ((f'(\varphi) - \dot{s}I)\tilde{w})_{\tilde{x}} + \delta\tilde{Q}(v^\varepsilon,\tilde{w})_{\tilde{x}} + (B\tilde{w})_{\tilde{x}} = \tilde{w}_{\tilde{x}\tilde{x}} + \tilde{F}_{\tilde{x}}. \quad (60)$$

Here $Q(v^\varepsilon,\delta\tilde{w}) = \delta^2\tilde{Q}(v^\varepsilon,\tilde{w})$, since Q is essentially quadratic in w.

In [10] a corresponding nonlinear estimate is proved by considering the omitted terms, $(Bw)_x$ and $\tilde{Q}_x$, as part of the forcing. Therefore we need to consider

$$\int_0^{\tilde{T}} \|Bw(\cdot,\tilde{t})\|^2_{\tilde{L}_{2,1}} d\tilde{t} + \big(\int_0^{\tilde{T}} \|Bw(\cdot,\tilde{t})\|_{\tilde{L}_{1,1}} d\tilde{t}\big)^2 \le \beta\int_0^{\tilde{T}} \|w(\cdot,\tilde{t})\|^2_{\tilde{L}_{2,1}})d\tilde{t},$$

$$\beta = |B|^2_\infty + |B_x|^2_\infty + \int_0^{\tilde{T}} \|B(\cdot,\tilde{t})\|^2_{\tilde{L}_{2,1}} d\tilde{t}.$$

The nonlinear term is estimated using its quadratic property and a Sobolev estimate for the maximum norm. From [10] we have the following theorem

Theorem 5.2 *If the assumptions 2.1, 2.2,2.3,2.4 are satisfied and δ and β are sufficiently small then the solution of (60) satisfies*

$$\int_0^{\tilde{T}} \|\tilde{w}(\cdot,\tilde{t})\|^2_{2,2} + \|\tilde{w}_t(\cdot,\tilde{t})\|^2_{2,1} d\tilde{t} \le 4RK. \quad (61)$$

Here R and K are the constants appearing in Lemma 5.1 and in (58), respectively.

The quantity δ can be made sufficiently small by choosing ε sufficiently small. The quantity β must also be sufficiently small. Therefore we make the following assumption

Assumption 5.3 *The initial data of the zeroth order term, (12), are constant states separated by a shock, that is*

$$g_0(x) = \begin{cases} u^+ & \textit{for } x > 0 \\ u^- & \textit{for } x < 0 \end{cases} \tag{62}$$

By assumption 2.2 $u^\pm$ together with v_0 satisfies the Rankine-Hugoniot condition. Clearly the zeroth order term, u_0, will be the constant states connected by a shock moving with speed v_0, that is

$$u_0(x,t) = \begin{cases} u^+ & \text{for } x > v_0 t \\ u^- & \text{for } x < v_0 t \end{cases}$$

It follows that the boundary condition for u_1 at the shock is homogeneous. Thus

$$\|u_1\|_{L_{2,1}} \leq K\alpha_1.$$

see (36). Further,

$$I(x,t) = \varphi_0(\frac{x - v_0 t}{\varepsilon}) + \mathcal{O}(\varepsilon) \quad O(x,t) = u_0(x,t) + \varepsilon u_1 + \mathcal{O}(\varepsilon^2). \tag{63}$$

Since $|d|$ is bounded, see Lemma 4.1, and φ_0 approaches its limiting values exponentially we have

$$|v^\varepsilon - \varphi_0|_\infty + |v^\varepsilon_{\tilde{x}} - \varphi_{0\tilde{x}}|_\infty \leq \varepsilon K.$$

Therefore $|B|_\infty + |B_x|_\infty \leq K\varepsilon$, and if ε is sufficiently small

$$\int_0^{T/\varepsilon}\int_{-\infty}^{\infty} |B|^2 d\tilde{x}d\tilde{t} \leq \int_0^{T/\varepsilon}\int_{-\infty}^{\infty} |\varphi_0 - v^\varepsilon|^2 d\tilde{x}d\tilde{t} \leq$$

$$C\{\int_0^{T/\varepsilon}\int_{|\tilde{x}|<\varepsilon^{\gamma-1}} (|d|^2 + \mathcal{O}(\varepsilon^2))d\tilde{x}d\tilde{t} + \int_0^{T/\varepsilon}\int_{|\tilde{x}|\geq\varepsilon^{\gamma-1}} (|\varphi_0 - u_0|^2 + |\varepsilon u_1 + \mathcal{O}(\varepsilon^2)|^2 + |d|^2)d\tilde{x}d\tilde{t}\}$$

$$\leq C(\varepsilon^\gamma + \frac{1}{\varepsilon}e^{-\varepsilon^{\gamma-1}} + \alpha_1^2) \leq K\alpha_1^2 \tag{64}$$

Here we have used (63) and (37). Similarly,

$$\int_0^{T/\varepsilon}\int_{-\infty}^{\infty} |B_{\tilde{x}}|^2 d\tilde{x}d\tilde{t} \leq \int_0^{T/\varepsilon}\int_{-\infty}^{\infty} C(|v^\varepsilon - \varphi|^2 + |v^\varepsilon_{\tilde{x}} - \varphi_{\tilde{x}}|^2)d\tilde{x}d\tilde{t} \leq$$

$$C_1\varepsilon^\gamma + \int_0^{T/\varepsilon}\int_{|\tilde{x}|\geq\varepsilon^{\gamma-1}} (|v^\varepsilon - \varphi|^2 + |v^\varepsilon_{\tilde{x}}|^2 + |\varphi_{\tilde{x}}|^2)d\tilde{x}d\tilde{t} \leq K\alpha_1^2. \tag{65}$$

Note that from (61) it follows that $|\tilde{w}(\tilde{t},\tilde{x})|$ is bounded uniformly. Thus we have the following theorem

Theorem 5.4 *If all the assumptions are satisfied and* α_1 *is sufficiently small then there exists constants* K *and* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \leq \varepsilon_0$

$$|u^{\varepsilon}(x,t) - v^{\varepsilon}(x,t)| \leq K\varepsilon^{4\gamma-2}, \quad -\infty < x < \infty,\, 0 < t < T.$$

Here K *is independent of* x, t *and* ε.

6 Conclusions

In this report we show that the solution of a slightly viscous conservation law can be approximated well by the first two terms in a matched asymptotic expansion.We prove the results for cases where the solution is close to a traveling wave. If a result corresponding to Lemma 5.1 was available where the Jacobian is evaluated along some more general solution than a traveling wave, this restriction could probably be removed.

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Report title

Approximate Solutions to Slightly Viscous Conservation Laws

Abstract

We study approximate solutions of a slightly viscous conservation law in one dimension, constructed by two asymptotic expansions that are cut off after the third order terms. In the shock layer an inner solution is valid and an outer solution is valid elsewhere. The two solutions are matched in a matching region.
Based on the stability results in [10] we show that for a given time interval the difference between the approximate solutions and the true solution is not larger than $\mathcal{O}(\varepsilon)$, where ε is the viscosity coefficient.

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Rapporttitel Approximativa lösningar till något viskösa konserveringslagar		

Sammanfattning

Approximativa lösningar till något viskösa konserveringslagar i en dimension studeras. Lösningarna konstrueras av två asymptotiska utvecklingar vilka avslutas efter den tredje ordningens term. En inre lösning är giltig inuti gränsskiktet kring stöten medan en yttre lösning är giltig i det övriga området. De två lösningarna kopplas ihop i ett matchningsområde.
Baserat på stabilitetsresultat som finns bevisat i [10], visar vi att för ett givet tidsintervall är skillnaden mellan de approximativa lösningarna och den riktiga lösningen inte större än $\mathcal{O}(\varepsilon)$, där ε är viskositetskoefficienten.

Nyckelord		
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