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Abstract

Multiple-scales technique (MSC) is used to examine the instability of non-parallel, compressible, quasi three-dimensional boundary layer flows. It models the kinematics and convective amplification of waves with weakly divergent or curved wave-rays and wave-fronts, propagating in a weakly non-uniform flow. The stability equations are put in a system of ordinary differential equations in a general orthogonal curvilinear coordinate system. The zeroth- order equations are homogeneous and govern the disturbance motion in a parallel flow and the non-local effects are calculated from the inhomogeneous first-order equations. The equations rewritten as a system of first order differential equations are discretized using compact finite difference scheme. For validation of the multiple-scales technique, we have compared the growth rates with results from 'parabolized stability equations' (PSE).

Contents

1 Introduction

The region where transition from laminar to turbulent flow occurs has to be determined when predicting skin friction and heat transfer along the surface of a body exposed to a flow field. The transition can be triggered by small unstable disturbances inside the boundary layer. The disturbances may be caused by local effects like surface roughness or are a response to external disturbances.

Laminar-turbulent transition is generally associated with an increase of energy losses due to increase of friction. This effect is more pronounced at supersonic speeds. The transition in the boundary layers on high speed vehicles both increases the surface heat transfer and affects the aerodynamic properties of such vehicles.

Most prediction methods used are either empirical or based on linear stability theory. The traditional stability theory, based on quasi parallel flow assumption, does not account for the growth of the boundary layer. The local character of this theory excludes any effects associated with the varying properties of the basic flow.

The non-parallel theory accounts for the weak dependence of the flow parameters on the streamwise coordinate, as well as the velocity normal to the wall. This theory will necessarily include the contribution of the streamwise distortion of the eigenfunction in the measurement of the growth rate of the disturbances. There are two major techniques to account for non-parallel effects. One is to solve the parabolic stability equations, see i.e. Hall [5], Itoh [10], Herbert & Bertolotti [9], Bertolotti [6] and Simen [14]. The other is the multiple-scales method, which was introduced into stability equations by Saric & Nayfeh [13], Gaponov [4] and El-Hady [3] among others.

In this presentation the method of multiple scales is used to model the instability of the compressible boundary layer. The zeroth order equations that corresponds to the parallel flow are put into a system of ordinary differential equations. The first order equations that includes the streamwise variation and non-parallel effects are calculated separately.

The purpose of this work is to develop a code to investigate the disturbance growth in compressible, quasi three-dimensional flows with the method of multiple-scales. To validate the calculations, test cases have been defined and compared with some PSE calculations.

2 Problem formulation

The stability equations are derived from the equations of conservation of mass, momentum and energy. The equation of state governing the flow of a viscous, compressible, ideal gas is formulated in primitive variables and in a general, curvilinear metric. The task is to model amplified waves with divergent or curved wave-rays and wave-fronts in a non-uniform flow. Wave amplitude, wave number and phase distribution can be spatially dependent if the flow is non-uniform. The non-dimensional conservation equations in vector notation are given by

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right] = -\nabla p + \frac{1}{R_0}\nabla[\lambda\,(\nabla\cdot\mathbf{u})] + \frac{1}{R_0}\nabla\cdot\left[\mu\,(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{tr}})\right], \tag{1}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{2}$$

$$\rho c_p\left[\frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)\,T\right] = \frac{1}{R_0 Pr}\nabla\cdot(\kappa\nabla T) + (\gamma-1)M^2\left[\frac{\partial p}{\partial t} + (\mathbf{u}\cdot\nabla)p + \frac{1}{R_0}\phi\right], \tag{3}$$

$$\gamma\mathbf{M}^2 p = \rho T \tag{4}$$

with viscous dissipation given as

$$\phi = \lambda(\nabla\cdot\mathbf{u})^2 + \frac{1}{2}\mu[\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{tr}}] : [\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{tr}}]. \tag{5}$$

Here, superscript ^{tr} refers to matrix transpose. Lengths, velocities and time are made dimensionless using a fixed reference length scale

$$l_0^* = \sqrt{\frac{\nu_0^* x_0^*}{U_0^*}}. \tag{6}$$

Here ρ,p,T stand for density, pressure and temperature, $\mathbf{u}$ is the velocity vector and t the time. The quantities λ and μ stand for the second and dynamic viscosity coefficient, γ is the ratio of specific heats, κ the heat conductivity, c_p the specific heat at constant pressure. U_0 is the local free-stream velocity and ν_0 is the kinematic viscosity coefficient and * refers to dimensional quantities. Mach number, M, Prandtl number, Pr and Reynolds number, R_0 are defined as

$$M = \frac{U_0^*}{\sqrt{\Re\gamma T_0^*}},$$

$$Pr = \frac{\mu_0^* c_{p0}^*}{\kappa_0^*}$$

and

$$R_0 = \frac{U_0^* l_0^*}{\nu_0^*},$$

where $\Re$ is the specific gas constant.
The mean flow quantities are weakly varying functions of the streamwise and spanwise coordinates, x^1 and x^2 respectively. The parameter ϵ characterizes this weak variation and dependence of the flow under study. Consider the compressible mean flow to be slightly non-parallel, that is the normal velocity component W is small compared to the streamwise component U. The derivatives of the scale factors, are also assumed to scale with the small parameter ϵ which is assumed to be of order O($1/R$). Then the slow scale can be introduced as $x_1^1 = \epsilon x^1$, $x_1^2 = \epsilon x^2$ and $t_1 = \epsilon t$. This scale governs the growth of the boundary layer, the modulation of the disturbance amplitude and the change in the eigenfunction. In the governing equations all flow and material quantities are decomposed into a steady basic flow plus an unsteady disturbance flow according to

$$\mathbf{q}(x_1^1, x_1^2, x^3, t_1) = \overline{\mathbf{q}}(x_1^1, x_1^2, x^3) + \widetilde{\mathbf{q}}(x_1^1, x_1^2, x^3, t_1) \tag{7}$$

where x^3 is normal to the directions of spatial amplification x^1 and x^2. Waves propagate in x^1, x^2 planes. Specifically in the above $\overline{\mathbf{q}}$ and $\widetilde{\mathbf{q}}$ stand for

$$\overline{\mathbf{q}} = [U, V, W, p, T, \rho], \tag{8}$$

$$\widetilde{\mathbf{q}} = [\widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{p}, \widetilde{T}, \widetilde{\rho}]. \tag{9}$$

Here U, V, W are the basic and $\widetilde{u}, \widetilde{v}, \widetilde{w}$ the disturbance velocity components in the streamwise, spanwise and normal directions, respectively. Assume that disturbance variables are divided into an amplitude function and an oscillating or wave function

$$\widetilde{\mathbf{q}} = \widehat{\mathbf{q}}(x_1^1, x_1^2, x^3, t_1) e^{i\theta} \tag{10}$$

where $\widehat{\mathbf{q}}$ stands for a complex, vectorial or scalar amplitude function and i for the imaginary unit. The complex phase function θ is defined such that

$$\frac{\partial \theta}{\partial x^1} = \alpha(x_1^1, x_1^2), \qquad \frac{\partial \theta}{\partial x^2} = \beta(x_1^1, x_1^2), \qquad \frac{\partial \theta}{\partial t} = -\omega. \tag{11}$$

The phase function is assumed continuously differentiable, thus

$$\frac{\partial \alpha}{\partial x_1^2} = \frac{\partial \beta}{\partial x_1^1} \tag{12}$$

α and β are the wavenumbers in the streamwise and spanwise directions respectively and ω is the frequency. In general these parameters are complex and given by $\alpha = \alpha_r + i\alpha_i$, $\beta = \beta_r + i\beta_i$ and $\omega = \omega_r + i\omega_i$, where the subscripts $_r$ and $_i$ refer to real and imaginary part of the quantities respectively. The physical components of the wave vector $\mathbf{k}(i)$ are defined as

$$\mathbf{k}(1) = \frac{\alpha_r}{h_1}, \qquad \mathbf{k}(2) = \frac{\beta_r}{h_2}, \qquad \mathbf{k}(3) = 0 \tag{13}$$

where h_i are the scale factors, see Appendix A.

3 Linearized stability equations

Here we follow the procedure used in the previous works by among others El-Hady[3] and Nayfeh [12]. To derive the stability equations an expansion in the form

$$\tilde{\mathbf{q}} = [\hat{\mathbf{q}}_0 + \epsilon\hat{\mathbf{q}}_1 + ...]e^{i\theta} \tag{14}$$

are made. We substitute equation (14) into equation (1)-(5) and collect coefficients of like powers of ϵ. Subtracting the mean flow equations and linearizing the equations, we obtain the zeroth- and first-order equations.

3.1 Zeroth-order equations

The zeroth-order equation system is written as

$$\mathcal{L}_0\hat{\mathbf{q}}_0 = 0. \tag{15}$$

where the zeroth-order operator $\mathcal{L}_0$ can be written as

$$\mathcal{L}_0 \equiv A + \frac{B}{h_3}\frac{\partial}{\partial x^3} + \frac{C}{h_3^2}\frac{\partial^2}{(\partial x^3)^2}. \tag{16}$$

The coefficients of the 5×5 matrices A, B and C are typed in Appendix A. Note that some terms of $O(1/R)$ are kept in the zeroth-order equations to remove the singularity near the critical points in the boundary layer where disturbance phase velocity is equal to the mean velocity. These are small everywhere and hence their effects can be neglected compared with the other terms which are $O(1)$. The zeroth-order equations are homogeneous and govern the disturbance motion in a parallel flow. In equation (15), $\hat{\mathbf{q}}_0$ stands for

$$\hat{\mathbf{q}}_0 = (\hat{u}_0, \hat{v}_0, \hat{T}_0, \hat{w}_0, \hat{\rho}_0)^{tr}. \tag{17}$$

The pressure disturbance is related to the density and temperature disturbances through the equation of state (4). The eighth-order ordinary differential operator $\mathcal{L}_0$ is subjected to the following boundary conditions

$$\hat{u}_0 = \hat{v}_0 = \hat{T}_0 = \hat{w}_0 = 0 \qquad at \qquad x^3 = 0 \tag{18}$$

$$\hat{u}_0 = \hat{v}_0 = \hat{T}_0 = \hat{w}_0 = 0 \qquad as \qquad x^3 \to \infty \tag{19}$$

The zeroth-order problem constitutes an eigenvalue problem whose solution provides the dispersion relation $\tau = \tau(\alpha, \beta, R_0, x_1^1, x_1^2, t_1)$ corresponding to the eigensolution

$$\hat{u}_0 = K(x_1^1, x_1^2, t_1)\xi_1(x_1^1, x_1^2, x^3) \tag{20}$$

$$\hat{v}_0 = K(x_1^1, x_1^2, t_1)\xi_2(x_1^1, x_1^2, x^3) \tag{21}$$

$$\hat{T}_0 = K(x_1^1, x_1^2, t_1)\xi_3(x_1^1, x_1^2, x^3) \tag{22}$$

$$\hat{w}_0 = K(x_1^1, x_1^2, t_1)\xi_4(x_1^1, x_1^2, x^3) \tag{23}$$

$$\hat{\rho}_0 = K(x_1^1, x_1^2, t_1)\xi_5(x_1^1, x_1^2, x^3) \tag{24}$$

where the function K is an amplitude modulation function and ξ_n are eigenfunctions. K is arbitary at this order.

3.2 First-order equations

The first-order equation system is written as

$$\mathcal{L}_0 \widehat{\mathbf{q}}_1 = \mathcal{F} \tag{25}$$

where $\mathcal{F} = \mathcal{F}(\mathcal{F}_c, \mathcal{F}_x, \mathcal{F}_y, \mathcal{F}_z, \mathcal{F}_e)^{tr}$ and $\widehat{\mathbf{q}}_1$ is the eigensolution of the first- order problem. The expressions for components of $\mathcal{F}$ are given in Appendix B.

The homogeneous part of the first-order equation is the same as the zeroth-order problem. The inhomogeneous problem (25) has a solution if, and only if, $\mathcal{F}$ is orthogonal to the solution of the adjoint of the eigenvalue problem (16). Here, the solvability condition is

$$\int_0^\infty (\mathcal{F}_c \xi_1^\dagger + \mathcal{F}_x \xi_2^\dagger + \mathcal{F}_y \xi_3^\dagger + \mathcal{F}_z \xi_4^\dagger + \mathcal{F}_e \xi_5^\dagger) dx^3 = 0, \tag{26}$$

where $\xi_n^\dagger$ are the solutions of the adjoint homogeneous problem which are defined in section 3.4. Now, the amplitude K of the zeroth-order solution can be determined by solving the equation (26).

The quantity $\mathcal{F}$ contains the spatial derivatives of the mean flow quantities and the eigenfunctions of the zeroth-order problem. Thus, it is necessary to evaluate the non-parallel terms $\partial \xi_n / \partial x_1^1$ and $\partial \xi_n / \partial x_1^2$. To do this, we replace the components of $\widehat{\mathbf{q}}_0$ in equation (16) with $\hat{u}_0 = \xi_1(x_1^1, x_1^2, x^3)$, $\hat{v}_0 = \xi_2(x_1^1, x_1^2, x^3)$, $\hat{T}_0 = \xi_3(x_1^1, x_1^2, x^3)$, $\hat{w}_0 = \xi_4(x_1^1, x_1^2, x^3)$, $\hat{\rho}_0 = \xi_5(x_1^1, x_1^2, x^3)$ and differentiate the equation with respect to x_1^1 and x_1^2. The inhomogeneous equation system that has to be solved is

$$\mathcal{L}_0 \frac{\partial \widehat{\mathbf{q}}_0}{\partial x_1^1} = \mathcal{G} \tag{27}$$

and

$$\mathcal{L}_0 \frac{\partial \widehat{\mathbf{q}}_0}{\partial x_1^2} = \mathcal{H}, \tag{28}$$

where $\mathcal{G} = (\mathcal{G}_c, \mathcal{G}_x, \mathcal{G}_y, \mathcal{G}_z, \mathcal{G}_e)^{tr}$ and $\mathcal{H} = (\mathcal{H}_c, \mathcal{H}_x, \mathcal{H}_y, \mathcal{H}_z, \mathcal{H}_e)^{tr}$, subject to the boundary conditions

$$\frac{\partial \xi_1}{\partial x_1^n} = \frac{\partial \xi_2}{\partial x_1^n} = \frac{\partial \xi_3}{\partial x_1^n} = \frac{\partial \xi_4}{\partial x_1^n} = 0 \qquad at \qquad x^3 = 0 \qquad (n = 1, 2) \tag{29}$$

$$\frac{\partial \xi_1}{\partial x_1^n} = \frac{\partial \xi_2}{\partial x_1^n} = \frac{\partial \xi_3}{\partial x_1^n} = \frac{\partial \xi_4}{\partial x_1^n} = 0 \qquad as \qquad x^3 \to \infty \qquad (n = 1, 2) \tag{30}$$

The homogeneous part of equations (27) and (28) has the same eigenvalues and adjoint as for the zeroth-order problem in equation (16). The quantities $\mathcal{G}$ and $\mathcal{H}$ contain spatial derivatives of α and β. To calculate these quantities, we apply the solvability condition to equations (27) and (28), which yield

$$\int_0^\infty (\mathcal{G}_c \xi_1^\dagger + \mathcal{G}_x \xi_2^\dagger + \mathcal{G}_y \xi_3^\dagger + \mathcal{G}_z \xi_4^\dagger + \mathcal{G}_e \xi_5^\dagger) dx^3 = 0 \tag{31}$$

and

$$\int_0^\infty (\mathcal{H}_c\xi_1^\dagger + \mathcal{H}_x\xi_2^\dagger + \mathcal{H}_y\xi_3^\dagger + \mathcal{H}_z\xi_4^\dagger + \mathcal{H}_e\xi_5^\dagger)dx^3 = 0. \tag{32}$$

Note that, in general, the equations above are not suficient to solve for all spatial derivatives of wavenumbers and further approximations should be made. This is out of scope of this paper. For a deeper discussion, the reader is refered to works of other authors, i.e. Nayfeh [12].

3.3 Group velocity and growth rate

Substituting the right hand side of equations (25) into equation (26) and rearranging, the following differential equation for the evolution of the amplitude K is obtained

$$g_1\frac{\partial K}{\partial t_1} + g_2\frac{\partial K}{\partial x_1^1} + g_3\frac{\partial K}{\partial x_1^2} = k_1 K \tag{33}$$

and

$$g_i = \int_0^\infty \widehat{g}_i dx^3, \qquad k_1 = \int_0^\infty \widehat{k}_1 dx^3 \tag{34}$$

where g_2/g_1 and g_3/g_1 are the components of the disturbance group velocity in the streamwise and spanwise directions, respectively, while k_1 reflects the effects of non-parallelism of the mean flow. The expressions for $\widehat{k}_1$ and $\widehat{g}_2$ are given in Appendix D. Similarly, we can rewrite the solvability equation (31) as

$$g_2\frac{\partial \alpha}{\partial x_1^1} + g_3\frac{\partial \beta}{\partial x_1^2} = k_2, \tag{35}$$

where

$$k_2 = \int_0^\infty \widehat{k}_2 dx^3. \tag{36}$$

The expression for $\widehat{k}_2$ is given in Appendix D.

To simplify the analysis we assume that both the basic and disturbance flow are independent of the x^2 coordinate. Introducing this simplification in equations (11) and (35) the following relations for α and β are found

$$\frac{\partial \alpha}{\partial x_1^1} = \frac{k_2}{g_2}. \tag{37}$$

$$\frac{\partial \beta}{\partial x_1^1} = 0, \tag{38}$$

Furthermore the imaginary part of the spanwise wavenumber is zero, i.e $\beta_i = 0$, and real part of it, β_r, is constant. The correction of growth rate can be obtained from equation (33). Here we consider spatial modulation of a single frequency disturbance, which means

$$\frac{\partial K}{\partial t_1} = 0. \tag{39}$$

Then, equation (33) simplifies to

$$\frac{dK}{dx_1^1} = i\tilde{\alpha}K \tag{40}$$

where

$$\tilde{\alpha} = -i\frac{k_1}{g_2}. \tag{41}$$

Therefore the amplitude K is

$$K = K_0 \exp{(i \int_{x_0^1}^{x^1} \epsilon\tilde{\alpha} dx^1)}. \tag{42}$$

K_0 is the initial amplitude at $x^1 = x_0^1$. Then the disturbance amplitude, to the first-order approximation, can be given as (10), (20)- (24) and (42) as

$$\tilde{\mathbf{q}}_n = K_0 \xi_n(x_1^1, x^3) \exp i\Big(\int (\alpha + \epsilon\tilde{\alpha})dx^1 + \beta x^2 - \omega t\Big). \tag{43}$$

Here the frequency ω is a real number and disturbances vary periodically only with x^2 and t. In contrast, their amplitude distribution varies in a non-periodic manner along the wave front, i. e. x^3-direction and x^1- direction if the flow is non-uniform or the rays are divergent. In equation (10), the physical disturbance is obtained from the real part of $\tilde{\mathbf{q}}$. In order to derive the physical wave number and amplification rate the components of the physical wave vector, which arises due to distortion of the eigenfunction, are introduced. A generalization of equation (13) can be defined as

$$k(1) = \frac{1}{h_1}\Big(-i\frac{1}{\tilde{\mathbf{q}}}\frac{\partial \tilde{\mathbf{q}}}{\partial x_1^1}\Big)_r, \qquad k(2) = \frac{1}{h_2}\Big(-i\frac{1}{\tilde{\mathbf{q}}}\frac{\partial \tilde{\mathbf{q}}}{\partial x_1^2}\Big)_r, \qquad k(3) = 0. \tag{44}$$

Equation (20)-(24) gives the physical disturbance amplitude as $K(x_1^1)\xi_n(x_1^1, x^3)$, where $K(x_1^1)$ is the amplitude modulation function and $\xi(x_1^1, x^3)$ is an eigenfunction. Part of ξ can be absorbed in K which means that $\tilde{\alpha}$ is dependent on the normalization of the eigenfunction ξ, see equation (40). The physical disturbance amplitude, on the other hand, is unique and independent of the eigenfunction normalization. The physical growth rate σ in direction of spatial amplification x^1 can be evaluated using

$$\sigma = -\alpha_i - \epsilon\tilde{\alpha}_i + \epsilon\Big(\frac{1}{\xi}\frac{\partial \xi}{\partial x_1^1}\Big)_r. \tag{45}$$

The first term in equation above is the parallel growth rate, while the second and the third terms together give the non-parallel correction.

The quantity ξ in (45) may be taken to be $\widehat{u}, \widehat{v}, \widehat{T}$ or $\widehat{\rho}U + \bar{\rho}\widehat{u}$, at some fixed x^3 or at the location where it reaches its maximum value. Note that this definition gives a growth rate which is a function of x^3. However, it is possible to use the integral of disturbance kinetic energy to meassure its growth rate.

$$E = \int_0^\infty \rho(|\widehat{u}|^2 + |\widehat{w}|^2 + |\widehat{v}|^2)dx^3 \tag{46}$$

Then, the growth rate is defined as

$$\sigma = -\alpha_i - \epsilon\tilde{\alpha}_i + \epsilon\frac{d}{dx_1^1}(\ln\sqrt{E}). \tag{47}$$

All growth rates presented here, use the definition given by equation (47).

3.4 Adjoint problem

$\mathcal{L}_0$ is a linear operator and we can define the adjoint operator $\mathcal{L}_0^\dagger$ as

$$(\widehat{\mathbf{q}}_0^\dagger, \mathcal{L}_0\widehat{\mathbf{q}}_0) = (\mathcal{L}_0^\dagger\widehat{\mathbf{q}}_0^\dagger, \widehat{\mathbf{q}}_0) \tag{48}$$

where $\widehat{\mathbf{q}}_0^\dagger$ is the adjoint eigensolution of the zeroth-order problem.The adjoint equations are derived using the definition of euclidian inner product

$$(\bar{u}, \bar{v}) = \int_0^\infty \bar{u}^H(x^3)\bar{v}(x^3)dx^3 \tag{49}$$

and integration by part. The adjoint homogeneous set of equations are written in the form

$$\mathcal{L}_0^\dagger\widehat{\mathbf{q}}_0^\dagger = A^\dagger\widehat{\mathbf{q}}_0^\dagger + B^\dagger\frac{1}{h_3}\frac{\partial\widehat{\mathbf{q}}_0^\dagger}{\partial x^3} + C^\dagger\frac{1}{h_3^2}\frac{\partial^2\widehat{\mathbf{q}}_0^\dagger}{(\partial x^3)^2} = 0 \tag{50}$$

$$A^\dagger = A^H - \frac{1}{h_3}\frac{\partial B^H}{\partial x^3} + \frac{1}{h_3^2}\frac{\partial h_3}{\partial x^3}B^H + \frac{1}{h_3^2}\frac{\partial^2 C^H}{(\partial x^3)^2} - \frac{4}{h_3^3}\frac{\partial h_3}{\partial x^3}\frac{\partial C^H}{\partial x^3}, \tag{51}$$

$$B^\dagger = -B^H + \frac{2}{h_3}\frac{\partial C^H}{\partial x^3} - \frac{4}{h_3^2}\frac{\partial h_3}{\partial x^3}C^H, \tag{52}$$

$$C^\dagger = C^H \tag{53}$$

where A^H, B^H and C^H are the transpose and complex conjugate of the zeroth-order coefficient matrices. The boundary conditions are

$$\xi_1^\dagger = \xi_2^\dagger = \xi_3^\dagger = \xi_4^\dagger = 0 \qquad at \qquad x^3 = 0 \tag{54}$$

$$\xi_1^\dagger = \xi_2^\dagger = \xi_3^\dagger = \xi_4^\dagger = 0 \qquad as \qquad x^3 \to \infty \tag{55}$$

4 Solution method

4.1 Numerical scheme

A fourth-order accurate compact finite difference scheme (see Lele [11]) is used to discretize the equation (16) and we write (16) as a system of eight first order differential equations in the following way

$$\mathcal{A}\psi + \mathcal{B}\frac{\partial\psi}{\partial x^3} = 0 \tag{56}$$

where

$$\psi = (\hat{u}_0, \hat{v}_0, \hat{T}_0, \hat{w}_0, \hat{\rho}_0, \hat{u}_0', \hat{v}_0', \hat{T}_0')^{tr} \tag{57}$$

where prime refers to the derivative with respect to the normal coordinate, x^3. The 8×8 matrices $\mathcal{A}$ and $\mathcal{B}$ can be represented as

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & -I \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} B & \frac{1}{h_3}\mathbf{C} \\ \mathbf{I} & 0 \end{pmatrix}$$

and I is a 3×3 identity matrix. Matrix $\mathbf{C}$ is a 5×3 matrix which contains the first three columns of the matrix C, and $\mathbf{I}$ is a 3×5 matrix with I as its left sub-matrix. When solving the equation system in equation (16), a fourth-order compact difference scheme is applied to approximate the derivatives of the normal coordinate. For this scheme the approximation on a non-uniform grid can be written (see [7])

$$c_1 f_{j+1} + c_2 f_j + c_3 f_{j-1} = C_1 f_{j+1}' + C_2 f_j' + C_3 f_{j-1}' \tag{58}$$

where f_j and f_j' are the values of the function and its derivative on the grid points, and

$$c_1 = 2(\triangle_{j-1})^3 \frac{2\triangle_j + \triangle_{j-1}}{(\triangle_j + \triangle_{j-1})}, \qquad c_2 = 2(\triangle_j + \triangle_{j-1})^2(\triangle_j - \triangle_{j-1}),$$

$$c_3 = -2(\triangle_j)^3 \frac{\triangle_j + 2\triangle_{j-1}}{(\triangle_j + \triangle_{j-1})},$$

$$C_1 = (\triangle_{j-1})^3\triangle_j, \qquad C_2 = (\triangle_j + \triangle_{j-1})^2\triangle_j\triangle_{j-1}, \qquad C_3 = (\triangle_j)^3\triangle_{j-1},$$

with $\triangle_j = x_{j+1}^3 - x_j^3$. For a uniform grid the coefficients in equation (58) reduce to

$$c_1 = -c_3 = \frac{3}{4\triangle_j}, \qquad c_2 = 0, \qquad C_1 = C_3 = \frac{1}{4}, \qquad C_2 = 1,$$

which recover the classical Padé scheme. Equation (58) can be represented in matrix form as

$$\mathcal{M}\mathbf{f} = \mathcal{N}\mathbf{f}' \tag{59}$$

where matrices $\mathcal{M}$ and $\mathcal{N}$ are tridiagonal (for more details see [7])

4.2 Discretized equations

The relation between the amplitude functions and their derivatives at the grid points is written as

$$\bar{\mathcal{M}}\hat{\psi} = \bar{\mathcal{N}}\frac{\partial \hat{\psi}}{\partial x^3} \tag{60}$$

where

$$\hat{\psi} = ((\hat{\psi}_0)^{tr}, ..., (\hat{\psi}_j)^{tr}, ..., (\hat{\psi}_N)^{tr})^{tr}, \tag{61}$$

and the subscript $_j$ denotes the j-th grid point in the normal direction. Matrices $\bar{\mathcal{M}}$ and $\bar{\mathcal{N}}$ have a block tridiagonal structure with c_1I, c_2I, c_3I and C_1I, C_2I, C_3I on the diagonals respectively. I is a 8×8 identity matrix. Inserting (60) into (56) yields

$$\bar{\mathcal{A}}\hat{\psi} + \bar{\mathcal{B}}\bar{\mathcal{N}}^{-1}\bar{\mathcal{M}}\hat{\psi} = 0, \tag{62}$$

which in the homogeneous case can be rewritten as

$$\mathcal{L}\hat{\psi} = 0, \tag{63}$$

and in the inhomogeneous case

$$\mathcal{L}\hat{\psi} = b, \tag{64}$$

where

$$\mathcal{L} = \left[\bar{\mathcal{N}}\bar{\mathcal{B}}^{-1}\bar{\mathcal{A}} + \bar{\mathcal{M}}\right],$$

and

$$b = -\bar{\mathcal{N}}\bar{\mathcal{B}}^{-1}\mathbf{f},$$

where $\mathbf{f}$ is the vector on the right hand side of the equation systems (25) and (27). Matrices $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ are block diagonal with matrices $\mathcal{A}$ and $\mathcal{B}$ as diagonal elements, see [7].

The equation system (56) is solved by using a partial pivoting Gauss elimination with backward substitutions method by using the Richtmyer algorithm, which is a fast solution algorithm for block- tridiagonal systems. For further details see [7].

4.3 Solution procedure

First the zeroth-order equations are solved to find α and $\widehat{\mathbf{q}}_0$. This is done in an iterative manner. One of the homogeneous boundary conditions, i.e. $\widehat{w}_0 = 0$ at $x^3 = 0$, is replaced by a normalization condition, i.e $\widehat{\rho}_0 = 1$ or $(\partial\widehat{u}_0/\partial x^3 = 1)$ at $x^3 = 0$. In each iteration step the value of α is updated using a secant method until the missing homogeneous boundary condition is fulfilled.

The correction of the growth rate due to the non-parallel effects are given by quantities k_1, g_2 which are functions of $\partial\widehat{q}_0/\partial x_1^1$. The latter one is given by the solution of the inhomogeneous equation (27). The fact that the operator $\mathcal{L}_0$ in (27) is the same as in (16), makes the problem singular. The matrix corresponding to the discretized equation has a sparse and block-tridiagonal form which makes use of a fast numerical algorithm possible. Here, the task is to disturb the operator matrix in order to avoid singularity without destroying its tridiagonal structure. In this case the use of Woodbury formula [15], which is the block-matrix version of the Sherman-Morrison formula, solves the problem. Woodburys formula is

$$(\widetilde{\mathcal{L}} + \mathbf{U} \cdot \mathbf{V}^H)^{-1} = \widetilde{\mathcal{L}}^{-1} - \left[\widetilde{\mathcal{L}}^{-1} \cdot \mathbf{U} \cdot (\mathbf{I} + \mathbf{V}^H \cdot \widetilde{\mathcal{L}}^{-1} \cdot \mathbf{U})^{-1} \cdot \mathbf{V}^H \cdot \widetilde{\mathcal{L}}^{-1}\right], \tag{65}$$

here $\widetilde{\mathcal{L}} = \mathcal{L} + e_1 \cdot e_1^{tr}$ is the disturbed $N \times N$ matrix and $e_1 = (1, 0, ..., 0)$ is the unit vector, while $\mathbf{U}$ and $\mathbf{V}$ are $N \times 2$ matrices. $\mathbf{U}$ is the matrix formed by columns out of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, and $\mathbf{V}$ is the matrix formed by columns out of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{u}_1$ and $\mathbf{v}_1$ are chosen to be the unit vector e_1 and $-e_1^{tr}$, respectively, $\mathbf{u}_2$ and $\mathbf{v}_2$ the null space out of $\mathcal{L}^\dagger$ and $\mathcal{L}$ respectively. Then the expressions for the correction to $\widetilde{\mathcal{L}}$ is

$$(\widetilde{\mathcal{L}} + \sum_{k=1}^{2} \mathbf{u}_k \otimes \mathbf{v}_k) = (\widetilde{\mathcal{L}} + \mathbf{U} \cdot \mathbf{V}^H) \tag{66}$$

Equation (65) is then solved in the following way:
first the problems

$$\widetilde{\mathcal{L}} \cdot \mathbf{z}_1 = \mathbf{u}_1, \qquad \widetilde{\mathcal{L}} \cdot \mathbf{z}_2 = \mathbf{u}_2, \tag{67}$$

are solved and the matrix

$$\mathbf{Z} \equiv [\mathbf{z}_1, \mathbf{z}_2] \tag{68}$$

obtained. Next, the 2×2 matrix inversion

$$\mathbf{H} \equiv (\mathbf{I} + \mathbf{V}^H \cdot \mathbf{Z})^{-1} \tag{69}$$

are done. $\mathbf{I}$ is the 2×2 identity matrix. The solution for $\widetilde{\mathcal{L}} \cdot \mathbf{y} = \mathbf{b}$, where $\mathbf{b}$ is the right hand side of equations (25) and (27), gives the solution vector $\mathbf{x}$

$$\mathbf{x} = \mathbf{y} - \mathbf{Z} \cdot \left[\mathbf{H} \cdot (\mathbf{V}^H \cdot \mathbf{y})\right]. \tag{70}$$

The solution of the first-order equations, due to singularity of $\mathcal{L}_0$, is not unique. To select the desiered solution we project the solution according to

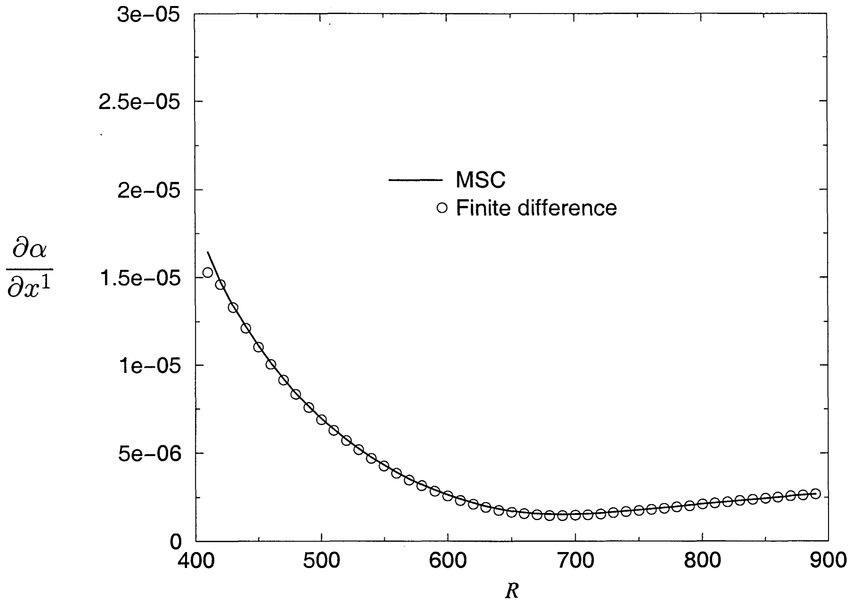
$$\hat{\mathbf{x}} = \mathbf{x} - (\mathbf{x}^T \cdot \mathbf{v}_2) \cdot \mathbf{v}_2 \tag{71}$$

where $\mathbf{v}_2$, which here is equal to $\widehat{\mathbf{q}}_0$, is normalized such that $\int_0^\infty \mathbf{v}_2^2 dx^3 = 1$. We also found an alternative way to remove the singularity of operator $\mathcal{L}_0$. Here we replace one of the homogeneous boundary conditions, i.e. $\partial \widehat{w}_0 / \partial x^1 = 0$ at $x^3 = 0$, with an normalization condition i.e. $\partial \widehat{p}_0 / \partial x^1 = 1$ at $x^3 = 0$. Both methods delivered same results.

5 Results and discussions

To check accuracy of the non-parallel terms $\partial\alpha/\partial x_1^1$ and $\partial\xi/\partial x_1^1$, the calculated values are compared with the finite difference approximation of the streamwise derivative of the wavenumber and eigensolutions of the zeroth-order problem.

Figure 1. derivative of α in streamwise direction.

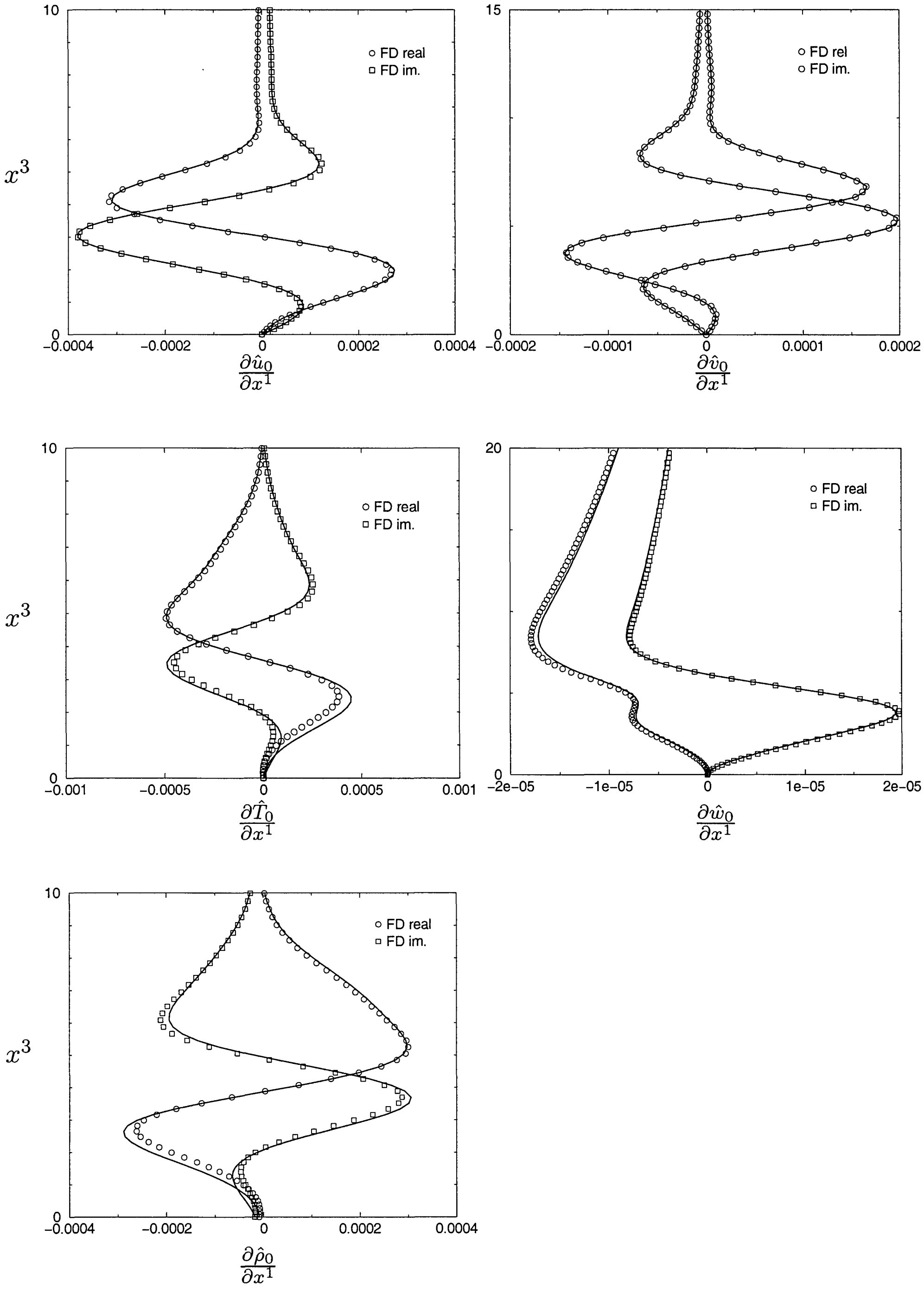


Comparisons of streamwise derivative of the wavenumber are presented in figure (1) and eigensolutions in figure (2). Here R is Reynolds number defined as $\frac{u_e^* l^*}{\nu_e^*}$, where $l^* = \sqrt{\frac{x_e^* \nu_e^*}{U_e^*}}$. According to the results all of the calculations are in good agreement with the solutions from finite differences. Here the Mach number is set to be 1.6 and the constant reduced frequency $F = 3 \times 10^{-4}$ and defined as

$$F = 2\pi f^* \frac{\nu_e^*}{U_e^{2*}} \tag{72}$$

where f^* is the constant, dimensional, physical frequency.
In the following sections the stability of boundary layer flows at different Mach numbers and for different geometries have been investigated. The growth rates are calculated based on the integral of disturbance kinetic energy and are made normalized using local reference length $\sqrt{\nu_e^* x^* / U_e^*}$.

Figure 2. derivatives of the eigensolutions compared with finite differences.



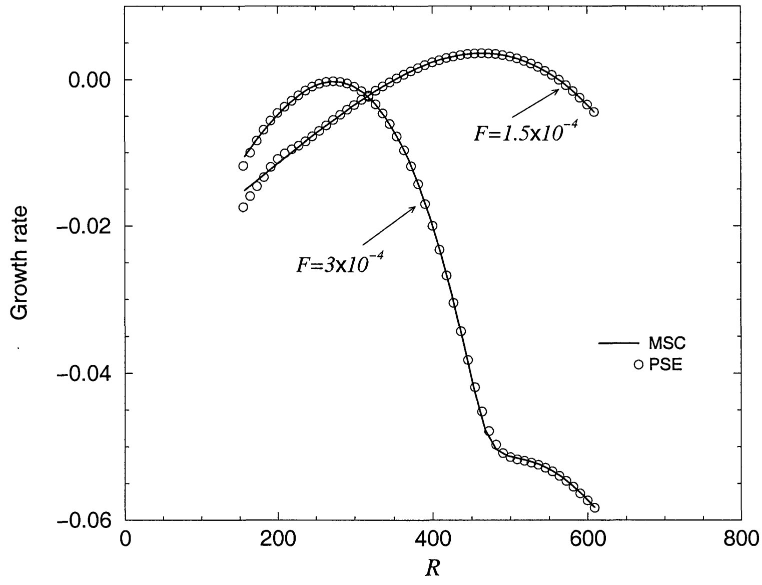


Figure 3. Variation of the spatial growth rates with Reynolds number in incompressible flat plate boundary layer flow of 2D waves ($M = 10^{-4}$).

5.1 Subsonic flow past a flat plate

The growth rates of disturbances in an incompressible flow are calculated using the multiple-scales (MSC) technique and compared with PSE. The Mach number was chosen to be 10^{-4} and calculations are made for two different reduced frequencies $F = 1.4 \times 10^{-4}$ and 3×10^{-4}. The growth rates are presented in figure (3). All of the results are in agreement with PSE except some small differences at initial part of calculation domain. This is due to the well known 'transient' behaviour of the PSE . The PSE calculations are started with solution of the zeroth-order equations.
Another test case for subsonic speeds were made for an oblique wave with $\beta/R = 1.5 \times 10^{-4}$, $F = 4 \times 10^{-5}$ and $M = 0.8$. This case is presented in figure (4) and shows excellent agreement with PSE data. Another test case models the flow over a swept flat plate with pressure gradient. The mean flow field is a Falkner-Skan-Cooke flow [2]. The mean velocity can be approximated $U_e = 14(x/c - 0.012)^{0.34207}$ m/s and $W_e = 14.15$ m/s, where $c = 0.5$. The dimensional frequency is $f^* = 158.17Hz$ and $\beta^* = 523.78$ m^{-1}. The result is plotted together with a stationary case in figure (5) and shows a good agreement with PSE data.

Figure 4. Variation of the spatial growth rates with Reynolds number at subsonic speeds in flat plate boundary layer flow of an oblique wave (M=0.8, $F = 4\times10^{-5}$, $\beta/R = 1.5\times10^{-4}$ and stagnation temperature $T_0 = 311$).

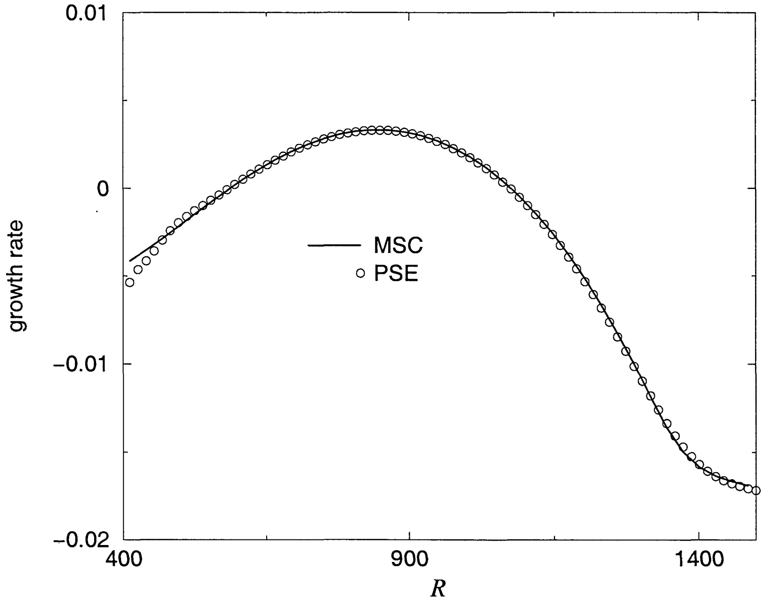
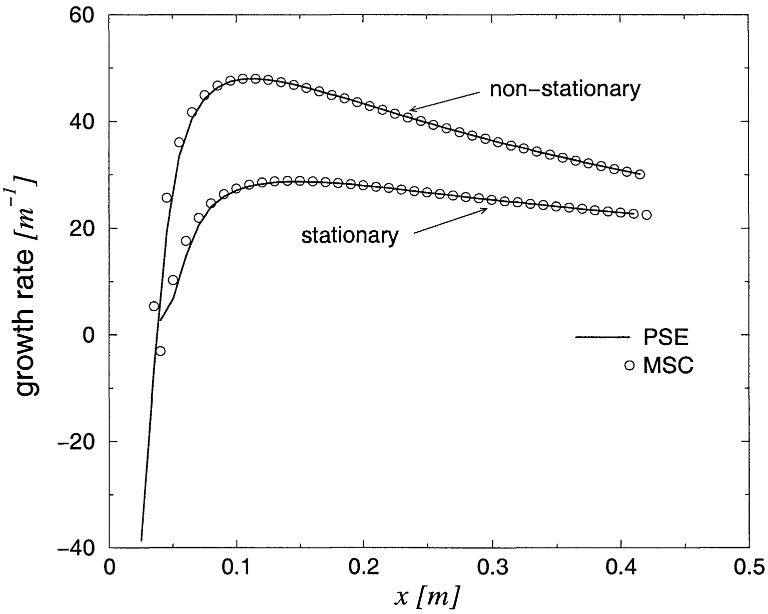


Figure 5. variation of growth rate of Falkner-Skan-Cooke velocity profiles (f= 158.17 Hz and $\beta = 523.78\ m^{-1}$).



5.2 Subsonic flow past a curved surface

To validate the implementation of the metric terms, we choose a two-dimensional subsonic boundary layer on a curved surface. The geometry of sinusoidal shaped plate, $y = sin(x)$, is used and both 2D and 3D disturbances are studied. The mean flow is given by self similar profiles.
The data are compared with PSE. The Mach number is $M = 0.8$, $\beta/R = 5.8 \times 10^{-5}$ and $F = 1 \times 10^{-5}$. Here, the results for all cases match those from PSE.

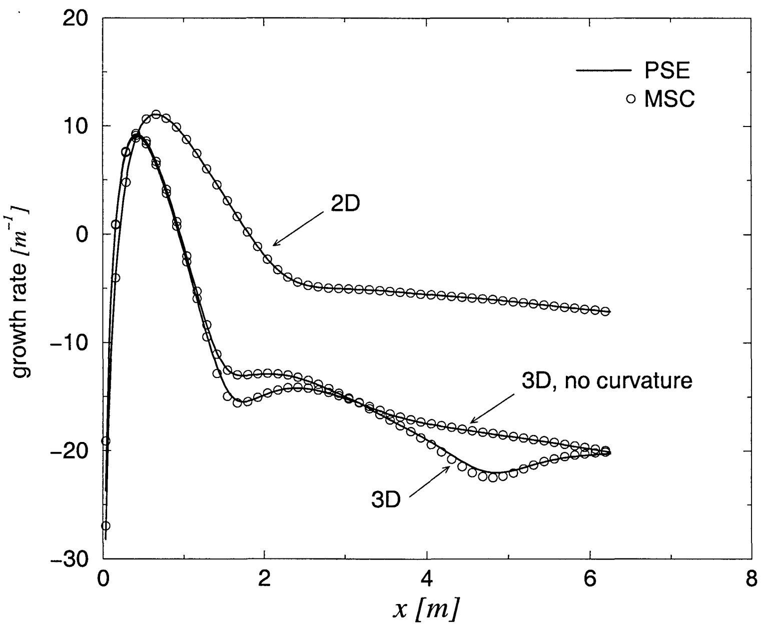


Figure 6. flow ower a sinus shaped plate ($M = 0.8$ and $\beta/R = 5.8 \times 10^{-5}$).

5.3 Subsonic flow past a swept wing

For more complicated basic flows we have selected a wing profile as the next test case. The geometry is an airfoil at 30 degrees sweep angle. The freestream Mach number is $M_\infty = 0.63$ and the Reynolds number based on the chord length is 8.5×10^6. The velocity at the edge of the boundary layer, Q_e, as a function of the arclength normal to the leading edge, S, is given in figure (7). To validate

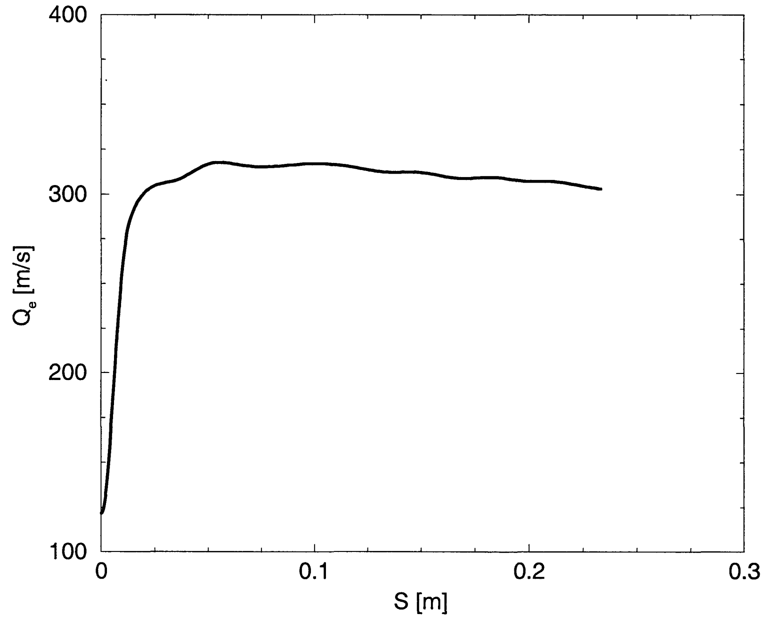


Figure 7. The velocity at the edge of the boundary layer, Q_e, as a function of the arclength normal to the leading edge, S ($M_\infty = 0.63$)

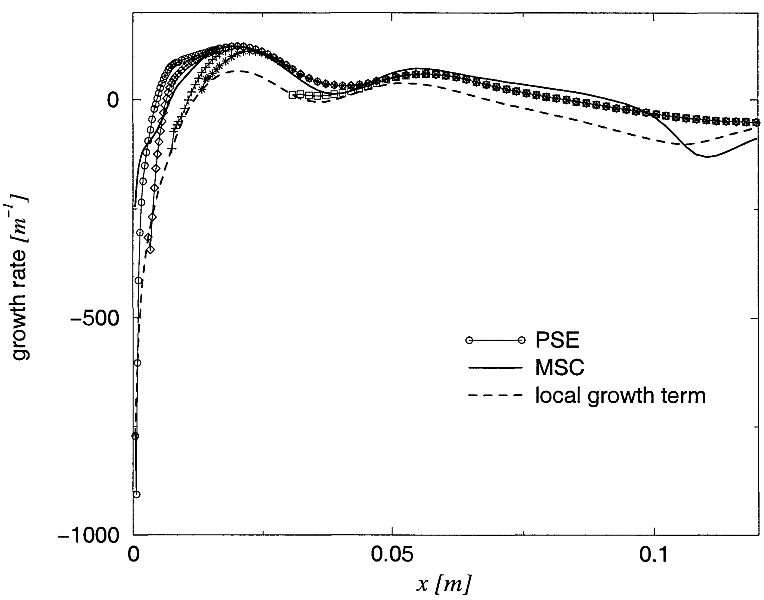


Figure 8. Variation of the spatial growth rates with Reynolds number at subsonic speed. Flow over a wing profile ($\beta = 5000\ m^{-1}$ and f* $= 16000\ Hz$)

the result, data are compared with PSE. The data are presented in figure (8). Five different start values for PSE are plotted. The local growth term is also presented to compare with. In this figure discrepancies from PSE can be detected. Here, all PSE results after a transient phase collapse to a single curve. The agreement between the two methods is not as good as in previous cases. The reason of this discrepancy is not found yet.

5.4 Supersonic flow past a flat plate

Tests were made at higher Mach numbers. The Mach number is chosen to be M=1.6. In figure (9) data are shown for PSE and MSC and data for the first two terms in equation (45) are also presented. They represents the parallel growth rate and nonparallel growth rate without the distortion effect of the eigenfunction, respectively. β/R is assumed to be 1.5×10^{-4} and the reduced frequency $F = 4 \times 10^{-5}$. The PSE calculations have been performed for different initial position. As is shown in figure (10) all the PSE reult after a interval of transient behaviour collapse to MSC results. The transient is larger when PSE calculations are started in the stable region.

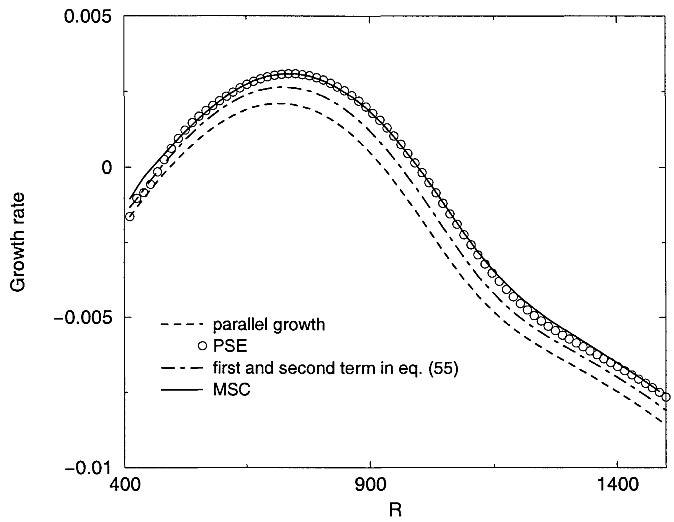


Figure 9. Variation of the spatial growth rates with Reynolds number at supersonic speeds (M=1.6, $\beta/R = 1.5 \times 10^{-4}$ and $F = 4 \times 10^{-5}$).

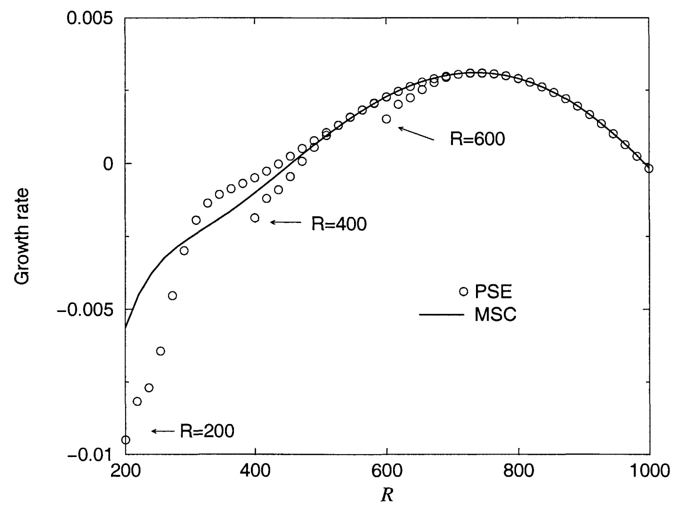


Figure 10. Variation of the spatial growth rates with Reynolds number at supersonic speeds for different start values (M=1.6, $\beta/R = 1.5 \times 10^{-4}$ and $F = 3 \times 10^{-5}$).

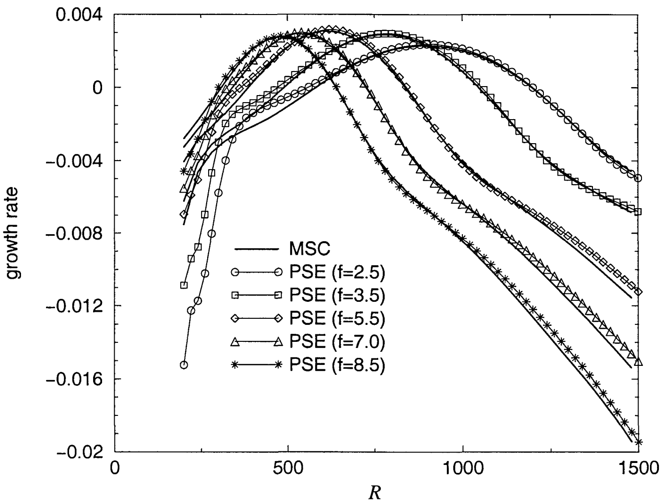


Figure 11. Variation of the spatial growth rates with Reynolds number at supersonic speeds and different frequencies ($M = 1.6$, $\beta/R = 1.5 \times 10^{-4}$ and f $= F \times 10^5$).

6 Conclusions

The method of multiple-scales is used to examine the non-parallel instability of compressible, quasi three-dimensional boundary layer flows. The method is applied to a flat plate and wing geometry. It models the kinematics and convective amplification of waves with weakly divergent or curved wave-rays and wavefronts, propagating in a weakly non-uniform flow. The stability equations, are put in a system of ordinary differential equations in a general, orthogonal curvilinear coordinates. The zeroth-order equations are homogeneous and govern the disturbance motion in a parallel flow and the nonlocal effects are calculated from the inhomogeneous terms of the first-order equations. The equations rewritten as a system of first order differential equations are discretized using compact finite difference scheme. Growth rates are calculated using an integral of disturbance kinetic energy. For validation of the multiple-scales technique, we have compared the growth rates with results from 'parabolized stability equations'(PSE). Discrepancies were detected for the flow over a airfoil at 30 degrees sweep angle. In general MSC and PSE results are in good agreement.

Acknowledgment

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Appendix A

Terms of order ϵ^0

The ortho1gonal curvilinear metric specifies the wave-rays and wave-fronts extending e.g. along a body contour. The first order problem contains metric quantities, i.e. the scale factors h_i, which yield the arc length along the rays and fronts,

$$h_i^2 = \sum_{j=1}^{3} (\frac{\partial y^j}{\partial x^i})^2 \tag{73}$$

and their derivatives, which yield divergence or curvature along the rays and fronts,

$$m_{ij} = \frac{1}{h_i h_j} \frac{\partial h_i}{\partial x^j}. \tag{74}$$

Here y^j are the Cartesian coordinates of the reference system.
Some definitions:

$$D_i = \frac{1}{h_i} \frac{\partial}{\partial x^i}, \qquad D_{ij} = \frac{1}{h_i h_j} \frac{\partial^2}{\partial x^i \partial x^j},$$

$$\alpha_0 = \frac{\alpha}{h_1}, \qquad \beta_0 = \frac{\beta}{h_2},$$

The homogeneous set of equation are written in the form

$$\mathcal{L}_0 \widehat{\mathbf{q}}_0 \equiv A\widehat{\mathbf{q}}_0 + B\frac{1}{h_3}\frac{\partial \widehat{\mathbf{q}}_0}{\partial x^3} + C\frac{1}{h_3^2}\frac{\partial^2 \widehat{\mathbf{q}}_0}{(\partial x^3)^2} = 0 \tag{75}$$

The non-zero components of the 5×5 matrices A, B and C are

$$\begin{aligned}
A(1,1) &= i\bar{\rho}\alpha_0 \\
A(1,2) &= i\beta_0\bar{\rho} \\
A(1,4) &= D_3\bar{\rho} \\
A(1,5) &= -i\omega + i\beta_0 V + i\alpha_0 U \\
A(2,1) &= i\bar{\rho}\alpha_0 U - i\bar{\rho}\omega + 2\frac{\mu\,\alpha_0{}^2}{R} + \frac{\lambda\alpha_0{}^2}{R} + \frac{\mu\,\beta_0{}^2}{R} + i\bar{\rho}\beta_0 V \\
A(2,2) &= \frac{\lambda\,\alpha_0\beta_0}{R} + \frac{\mu\alpha_0\beta_0}{R} \\
A(2,3) &= -\frac{D_3\,U\,\frac{1}{\mu}\frac{d^2\mu}{dT^2}\,D_3\,T}{R} - \frac{\frac{1}{\mu}\frac{d\mu}{dT}\,D_{33}\,U}{R} + i\alpha_0\frac{1}{\gamma M^2}\,\bar{\rho} \\
A(2,4) &= \frac{-i\alpha_0\frac{1}{\mu}\frac{d\mu}{dT}\,D_3\,T}{R} + \bar{\rho}\,D_3\,U \\
A(2,5) &= i\alpha_0\frac{1}{\gamma M^2}\,T \\
A(3,1) &= \frac{\lambda\,\alpha_0\beta_0}{R} + \frac{\mu\,\alpha_0\beta_0}{R}
\end{aligned}$$

$$
\begin{aligned}
A(3,2) &= \frac{\mu\,\alpha_0{}^2}{R} + \frac{\lambda\,\beta_0{}^2}{R} + 2\,\frac{\mu\,\beta_0{}^2}{R} + i\bar\rho\,\beta_0 V + i\bar\rho\,\alpha_0 U - i\bar\rho\,\omega \\
A(3,3) &= -\frac{D_3\,V\,\frac{1}{\mu}\frac{d^2\mu}{dT^2}\,D_3\,T}{R} - \frac{\frac{1}{\mu}\frac{d\mu}{dT}\,D_{33}\,V}{R} + i\beta_0\frac{1}{\gamma M^2}\,\bar\rho \\
A(3,4) &= \bar\rho\,D_3\,V - \frac{i\beta_0\frac{1}{\mu}\frac{d\mu}{dT}\,D_3\,T}{R} \\
A(3,5) &= i\beta_0\frac{1}{\gamma M^2}\,T \\
A(4,1) &= \frac{-i\frac{\lambda}{\mu}\,\frac{1}{\mu}\frac{d\mu}{dT}\,D_3\,T\,\alpha_0}{R} + \frac{2\,i\mu\,D_3\bar\rho\,\alpha_0}{R\bar\rho} + \frac{i\frac{\lambda}{\mu}\,\mu\,D_3\bar\rho\,\alpha_0}{R\bar\rho} \\
A(4,2) &= \frac{2\,i\mu\,\beta_0 D_3\bar\rho}{R\bar\rho} - \frac{i\frac{\lambda}{\mu}\,\frac{1}{\mu}\frac{d\mu}{dT}\,D_3\,T\,\beta_0}{R} + \frac{i\lambda\,\beta_0 D_3\bar\rho}{R\bar\rho} \\
A(4,3) &= \frac{1}{\gamma M^2}\,D_3\bar\rho - \frac{i\frac{1}{\mu}\frac{d\mu}{dT}\,\beta_0 D_3\,V}{R} - \frac{iD_3\,U\,\frac{1}{\mu}\frac{d\mu}{dT}\,\alpha_0}{R} \\
A(4,4) &= \frac{\mu\,\beta_0{}^2}{R} + \frac{\lambda\,D_{33}\bar\rho}{R\bar\rho} + i\bar\rho\,\alpha_0 U - i\bar\rho\,\omega + i\bar\rho\,\beta_0 V + 2\,\frac{\mu\,D_{33}\bar\rho}{R\bar\rho} + \\
&\quad \frac{\mu\,\alpha_0{}^2}{R} \\
A(4,5) &= \frac{2\,i\mu\,\alpha_0 D_3\,U}{R\bar\rho} + \frac{1}{\gamma M^2}\,D_3\,T + \frac{i\lambda\,\alpha_0 D_3\,U}{R\bar\rho} + \frac{2\,i\mu\,\beta_0 D_3\,V}{R\bar\rho} + \\
&\quad \frac{i\frac{\lambda}{\mu}\,\mu\,\beta_0 D_3\,V}{R\bar\rho} \\
B(1,4) &= \bar\rho \\
B(2,1) &= -\frac{\frac{1}{\mu}\frac{d\mu}{dT}\,D_3\,T}{R} \\
B(2,3) &= -\frac{D_3\,U\,\frac{1}{\mu}\frac{d\mu}{dT}}{R} \\
B(2,4) &= \frac{-i\lambda\,\alpha_0}{R} - \frac{i\mu\,\alpha_0}{R} \\
B(3,2) &= -\frac{\frac{1}{\mu}\frac{d\mu}{dT}\,D_3\,T}{R} \\
B(3,3) &= -\frac{D_3\,V\,\frac{1}{\mu}\frac{d\mu}{dT}}{R} \\
B(3,4) &= \frac{-i\lambda\,\beta_0}{R} - \frac{i\mu\,\beta_0}{R} \\
B(4,1) &= \frac{i\mu\,\alpha_0}{R} \\
B(4,2) &= \frac{i\mu\,\beta_0}{R} \\
B(4,3) &= \frac{1}{\gamma M^2}\,\bar\rho \\
B(4,4) &= -2\,\frac{\frac{1}{\mu}\frac{d\mu}{dT}\,D_3\,T}{R} - \frac{\frac{\lambda}{\mu}\,\frac{1}{\mu}\frac{d\mu}{dT}\,D_3\,T}{R} + 4\,\frac{\mu\,D_3\bar\rho}{R\bar\rho} + 2\,\frac{\lambda\,D_3\bar\rho}{R\bar\rho} \\
B(4,5) &= \frac{-2\,i\mu\,\omega}{R\bar\rho} + \frac{1}{\gamma M^2}\,T - \frac{i\lambda\,\omega}{R\bar\rho} + \frac{i\lambda\,\alpha_0 U}{R\bar\rho} + \frac{i\lambda\,\beta_0 V}{R\bar\rho} + \\
&\quad \frac{2\,i\mu\,\alpha_0 U}{R\bar\rho} + \frac{2\,i\mu\,\beta_0 V}{R\bar\rho}
\end{aligned}
$$

$$
\begin{aligned}
C(2,1) &= -\frac{\mu}{R} \\
C(3,2) &= -\frac{\mu}{R} \\
C(5,3) &= \frac{\kappa}{R\,\mathrm{Pr}}
\end{aligned}
$$

Appendix B

Forcing terms in first-order equations

We define first order equations $\mathcal{L}_1 \widehat{\mathrm{q}}_1 = \mathcal{F}$ where $\mathcal{F} = (\mathcal{F}_c, \mathcal{F}_x, \mathcal{F}_y, \mathcal{F}_z, \mathcal{F}_e)^{tr}$

$$\begin{aligned}\mathcal{F}_c \;=\; & (\widehat{\rho}_0)\; D_1 U + m_{13}\rho\,\widehat{w}_0 + m_{23}\rho\,\widehat{w}_0 + m_{21}\,(\widehat{\rho}_0)\; U + m_{21}\rho\,\widehat{u}_0 + \\ & (\widehat{\rho}_0)\; D_3 W + D_1 \rho\,\widehat{u}_0 + \rho\, D_1 \widehat{u}_0 + D_1 \widehat{\rho}_0\, U + m_{31}\,(\widehat{\rho}_0)\; U + \\ & m_{31}\rho\,\widehat{u}_0 + D_3 \widehat{\rho}_0\, W\end{aligned}$$

$$\begin{aligned}\mathcal{F}_x \;=\; & \frac{1}{\gamma M^2} D_1 \widehat{\rho}_0\, T - (\widehat{\rho}_0)\; m_{21} V^2 + \rho\, D_1\, U\,\widehat{u}_0 + \frac{1}{\gamma M^2}\,(\widehat{\rho}_0)\; D_1\, T - \\ & 2\,\rho\, m_{21} V \widehat{v}_0 + \rho\, D_1 \widehat{u}_0\, U + \rho\, m_{13} \widehat{w}_0\, U + \rho\, D_3 \widehat{u}_0\, W + \\ & (\widehat{\rho}_0)\; D_1\, U\, U + (\widehat{\rho}_0)\; D_3\, U\, W + \frac{1}{\gamma M^2}\,\rho\, D_1\, \widehat{T}_0 + \frac{1}{\gamma M^2} D_1 \rho\, \widehat{T}_0\end{aligned}$$

$$\begin{aligned}\mathcal{F}_y \;=\; & \rho\, m_{21} \widehat{u}_0\, V + \rho\, m_{21} U \widehat{v}_0 + \rho\, m_{23} \widehat{w}_0\, V + (\widehat{\rho}_0)\; m_{21} U V + \\ & (\widehat{\rho}_0)\; D_1\, V\, U + \rho\, D_1 \widehat{v}_0\, U + \rho\, D_1\, V\, \widehat{u}_0 + \\ & \rho\, D_3 \widehat{v}_0\, W + (\widehat{\rho}_0)\; D_3\, V\, W\end{aligned}$$

$$\begin{aligned}\mathcal{F}_z \;=\; & -(\widehat{\rho}_0)\; m_{13} U^2 + \rho\, m_{31} U \widehat{w}_0 - (\widehat{\rho}_0)\; m_{23} V^2 + \rho\, D_1 \widehat{w}_0\, U + \\ & \rho\, D_3\, W\, \widehat{w}_0 + \rho\, D_3 \widehat{w}_0\, W - 2\,\rho\, m_{23} V \widehat{v}_0 - 2\,\rho\, m_{13} U \widehat{u}_0\end{aligned}$$

$$\begin{aligned}\mathcal{F}_e \;=\; & -\rho\, cp\, W D_3\, \widehat{T}_0 + (\gamma - 1)\, U \frac{1}{\gamma}\, D_1 \widehat{\rho}_0\, T - \rho\, cp\, \widehat{u}_0\, D_1\, T - \\ & \rho\, \frac{1}{c_p} \frac{dc_p}{dT}\, \widehat{T}_0\, W D_3\, T - \rho\, \frac{1}{c_p} \frac{dc_p}{dT}\, \widehat{T}_0\, U D_1\, T + \\ & (\gamma - 1)\, U \frac{1}{\gamma}\, (\widehat{\rho}_0)\; D_1\, T + (\gamma - 1)\, U \frac{1}{\gamma}\, D_1 \rho\, \widehat{T}_0 + \\ & (\gamma - 1)\, U \frac{1}{\gamma}\, \rho\, D_1\, \widehat{T}_0 + (\gamma - 1)\, W \frac{1}{\gamma}\, D_3 \widehat{\rho}_0\, T + \\ & (\gamma - 1)\, W \frac{1}{\gamma}\, (\widehat{\rho}_0)\; D_3\, T + (\gamma - 1)\, W \frac{1}{\gamma}\, D_3 \rho\, \widehat{T}_0 + \\ & (\gamma - 1)\, W \frac{1}{\gamma}\, \rho\, D_3\, \widehat{T}_0 + (\gamma - 1)\, M^2\, \widehat{u}_0\, D_1 P - \\ & (\widehat{\rho}_0)\; cp\, W D_3\, T - (\widehat{\rho}_0)\; cp\, U D_1\, T - \rho\, cp\, U D_1\, \widehat{T}_0\end{aligned}$$

Appendix C

Forcing terms in equations for $\frac{\partial \widehat{q}_0}{\partial x_1^1}$

We define equations $\mathcal{L}_0 \frac{\partial \widehat{q}_0}{\partial x^1} = \mathcal{G}$ as where $\mathcal{G} = (\mathcal{G}_c, \mathcal{G}_x, \mathcal{G}_y, \mathcal{G}_z, \mathcal{G}_e)$

$$
\begin{aligned}
\frac{\mathcal{G}_c}{h_1} = {} & -D_3 \rho\, \widehat{w}_0\, m_{31} + D_{13} \rho\, \widehat{w}_0 + i\alpha_0 \left(\widehat{\rho}_0\right) D_1 U - \\
& i\beta_0 \rho\, \widehat{v}_0\, m_{21} + i\beta_0 D_1 \rho\, \widehat{v}_0 - i\beta_0 \left(\widehat{\rho}_0\right) V m_{21} + i\rho \frac{1}{h_1^2} \frac{\partial \alpha}{\partial x^1}\, \widehat{u}_0 + \\
& i\beta_0 \left(\widehat{\rho}_0\right) D_1 V - i\alpha_0 \left(\widehat{\rho}_0\right) U m_{11} + i \frac{1}{h_1^2} \frac{\partial \alpha}{\partial x^1} \left(\widehat{\rho}_0\right) U - \rho\, D_3 \widehat{w}_0\, m_{31} + \\
& D_1 \rho\, D_3 \widehat{w}_0 - i\rho\, \alpha_0 \widehat{u}_0\, m_{11} + Im D_1 \rho\, \alpha_0 \widehat{u}_0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathcal{G}_x}{h_1} = {} & i\rho\, \alpha_0 \widehat{u}_0\, D_1 U - i\rho\, \alpha_0 \widehat{u}_0\, U m_{11} + \rho\, D_{13} U\, \widehat{w}_0 + i D_1 \rho\, \alpha_0 \widehat{u}_0\, U + \\
& i\rho \frac{1}{h_1^2} \frac{\partial \alpha}{\partial x^1}\, \widehat{u}_0\, U + D_1 \rho\, D_3 U\, \widehat{w}_0 - i\alpha_0 \frac{1}{\gamma M^2} \left(\widehat{\rho}_0\right) T m_{11} + \\
& i \frac{1}{h_1^2} \frac{\partial \alpha}{\partial x^1} \frac{1}{\gamma M^2} \left(\widehat{\rho}_0\right) T + i\alpha_0 \frac{1}{\gamma M^2} D_1 \rho\, \widehat{T}_0 - i\alpha_0 \frac{1}{\gamma M^2} \rho\, \widehat{T}_0\, m_{11} + \\
& i D_1 \rho\, \beta_0 \widehat{u}_0\, V + i \frac{1}{h_1^2} \frac{\partial \alpha}{\partial x^1} \frac{1}{\gamma M^2} \rho\, \widehat{T}_0 + i\alpha_0 \frac{1}{\gamma M^2} \left(\widehat{\rho}_0\right) D_1 T - \\
& i\rho\, \beta_0 \widehat{u}_0\, V m_{21} + i\rho\, \beta_0 \widehat{u}_0\, D_1 V - i D_1 \rho\, \omega\, \widehat{u}_0 - \rho\, D_3 U\, \widehat{w}_0\, m_{31}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathcal{G}_y}{h_1} = {} & i\rho \frac{1}{h_1^2} \frac{\partial \alpha}{\partial x^1}\, \widehat{v}_0\, U + i D_1 \rho\, \alpha_0 \widehat{v}_0\, U - i D_1 \rho\, \omega\, \widehat{v}_0 + i D_1 \rho\, \beta_0 \widehat{v}_0\, V + \\
& i\beta_0 \frac{1}{\gamma M^2} D_1 \rho\, \widehat{T}_0 + i\beta_0 \frac{1}{\gamma M^2} \left(\widehat{\rho}_0\right) D_1 T - i\beta_0 \frac{1}{\gamma M^2} \rho\, \widehat{T}_0\, m_{21} + \\
& \rho\, D_{13} V\, \widehat{w}_0 - i\rho\, \beta_0 \widehat{v}_0\, V m_{21} - \rho\, D_3 V\, \widehat{w}_0\, m_{31} + D_1 \rho\, D_3 V\, \widehat{w}_0 - \\
& i\rho\, \alpha_0 \widehat{v}_0\, U m_{11} + i\rho\, \alpha_0 \widehat{v}_0\, D_1 U + i\rho\, \beta_0 \widehat{v}_0\, D_1 V - i\beta_0 \frac{1}{\gamma M^2} \left(\widehat{\rho}_0\right) T m_{21}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathcal{G}_z}{h_1} = {} & -\frac{1}{\gamma M^2} \rho\, D_3 \widehat{T}_0\, m_{31} + i D_1 \rho\, \alpha_0 \widehat{w}_0\, U + i\rho \frac{1}{h_1^2} \frac{\partial \alpha}{\partial x^1}\, \widehat{w}_0\, U - \\
& \frac{1}{\gamma M^2} D_3 \widehat{\rho}_0\, T m_{31} + \frac{1}{\gamma M^2} D_1 \rho\, D_3 \widehat{T}_0 + i D_1 \rho\, \beta_0 \widehat{w}_0\, V + \\
& i\rho\, \beta_0 \widehat{w}_0\, D_1 V - i\rho\, \beta_0 \widehat{w}_0\, V m_{21} - i D_1 \rho\, \omega\, \widehat{w}_0 - i\rho\, \alpha_0 \widehat{w}_0\, U m_{11} - \\
& \frac{1}{\gamma M^2} D_3 \rho\, \widehat{T}_0\, m_{31} - \frac{1}{\gamma M^2} \left(\widehat{\rho}_0\right) D_3 T\, m_{31} + \frac{1}{\gamma M^2} \left(\widehat{\rho}_0\right) D_{13} T + \\
& \frac{1}{\gamma M^2} D_3 \widehat{\rho}_0\, D_1 T + \frac{1}{\gamma M^2} D_{13} \rho\, \widehat{T}_0 + i\rho\, \alpha_0 \widehat{w}_0\, D_1 U
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathcal{G}_e}{h_1} \;=\; & -i\rho\, cp\, U \frac{1}{h_1^2}\frac{\partial\alpha}{\partial x^1}\,\widehat{T}_0 + i\rho\, cp\, U\alpha_0\widehat{T}_0\, m_{11} - iD_1\rho\, cp\, U\alpha_0\widehat{T}_0 - \\
& \rho\, cp\,\widehat{w}_0\, D_{13}\, T + i(\gamma-1)\, U\frac{1}{h_1^2}\frac{\partial\alpha}{\partial x^1}\,\frac{1}{\gamma}\,\rho\,\widehat{T}_0 + \\
& i(\gamma-1)\,\beta_0 V\frac{1}{\gamma}\,(\widehat{\rho}_0)\; D_1\, T - i\rho\, cp\,\beta_0 D_1\, V\,\widehat{T}_0 + \\
& i\rho\,\frac{1}{c_p}\frac{dc_p}{dT}\, D_1\, T\,\omega\,\widehat{T}_0 - i\rho\,\frac{1}{c_p}\frac{dc_p}{dT}\, D_1\, T\,\beta_0 V\,\widehat{T}_0 - \\
& i(\gamma-1)\,\beta_0 V\frac{1}{\gamma}\,\rho\,\widehat{T}_0\, m_{21} + (\gamma-1)\, M^2\,\widehat{w}_0\, D_{13} P + \\
& \rho\, cp\,\widehat{w}_0\, D_3\, T\, m_{31} - (\gamma-1)\, M^2\,\widehat{w}_0\, D_3 P\, m_{31} + \\
& i(\gamma-1)\, D_1\, U\,\alpha_0\frac{1}{\gamma}\,(\widehat{\rho}_0)\; T - i(\gamma-1)\,\omega\,\frac{1}{\gamma}\,(\widehat{\rho}_0)\; D_1\, T + \\
& i(\gamma-1)\,\beta_0 V\frac{1}{\gamma}\, D_1\rho\,\widehat{T}_0 - D_1\rho\, cp\,\widehat{w}_0\, D_3\, T - \\
& i\rho\,\frac{1}{c_p}\frac{dc_p}{dT}\, D_1\, T\, U\alpha_0\widehat{T}_0 - \\
& \rho\,\frac{1}{c_p}\frac{dc_p}{dT}\, D_1\, T\,\widehat{w}_0\, D_3\, T + i\rho\, cp\,\beta_0 V\,\widehat{T}_0\, m_{21} + \\
& i(\gamma-1)\, U\alpha_0\frac{1}{\gamma}\, D_1\rho\,\widehat{T}_0 + iD_1\rho\, cp\,\omega\,\widehat{T}_0 - \\
& i(\gamma-1)\,\omega\,\frac{1}{\gamma}\, D_1\rho\,\widehat{T}_0 - i\rho\, cp\, D_1\, U\,\alpha_0\widehat{T}_0 - \\
& i(\gamma-1)\, U\alpha_0\frac{1}{\gamma}\,\rho\,\widehat{T}_0\, m_{11} + i(\gamma-1)\, U\frac{1}{h_1^2}\frac{\partial\alpha}{\partial x^1}\,\frac{1}{\gamma}\,(\widehat{\rho}_0)\; T + \\
& i(\gamma-1)\,\beta_0 D_1\, V\,\frac{1}{\gamma}\,\rho\,\widehat{T}_0 - i(\gamma-1)\, U\alpha_0\frac{1}{\gamma}\,(\widehat{\rho}_0)\; T m_{11} + \\
& i(\gamma-1)\, U\alpha_0\frac{1}{\gamma}\,(\widehat{\rho}_0)\; D_1\, T - iD_1\rho\, cp\,\beta_0 V\,\widehat{T}_0 + \\
& i(\gamma-1)\, D_1\, U\,\alpha_0\frac{1}{\gamma}\,\rho\,\widehat{T}_0 - i(\gamma-1)\,\beta_0 V\frac{1}{\gamma}\,(\widehat{\rho}_0)\; T m_{21} + \\
& i(\gamma-1)\,\beta_0 D_1\, V\,\frac{1}{\gamma}\,(\widehat{\rho}_0)\; T
\end{aligned}
$$

Appendix D

Coefficients of disturbance amplitude equation

We define some terms to integrate in the solvability condition.

$$
\begin{aligned}
\widehat{g}_2 \;=\;& \xi^*{}_1 U\xi_5 + \xi^\dagger{}_1\rho\,\xi_1 + \xi^\dagger{}_2\frac{1}{\gamma M^2}\,T\xi_5 + \xi^\dagger{}_2\rho\,U\xi_1 + \xi^\dagger{}_2\frac{1}{\gamma M^2}\,\rho\,\xi_3 + \\
& \xi^\dagger{}_3\rho\,U\xi_2 + \xi^\dagger{}_4\rho\,U\xi_4 + \xi^\dagger{}_5(\gamma-1)\,U\frac{1}{\gamma}\,\rho\,\xi_3 + \xi^\dagger{}_5(\gamma-1)\,U\frac{1}{\gamma}\,T\xi_5 - \\
& \xi^\dagger{}_5\rho\,cp\,U\xi_3
\end{aligned}
$$

$$
\begin{aligned}
\widehat{k}_1 \;=\;& i\xi^\dagger{}_1 m_{21}\xi_5 U + i\xi^\dagger{}_1 D_1\rho\,\xi_1 + i\xi^\dagger{}_1\xi_5 D_1\,U + i\xi^\dagger{}_3\rho\,m_{21}U\xi_2 + \\
& i\xi^\dagger{}_3\rho\,U D_1\xi_2 + i\xi^\dagger{}_3\rho\,m_{21}\xi_1 V + i\xi^\dagger{}_3\rho\,D_1\,V\,\xi_1 + \\
& i\xi^\dagger{}_3\rho\,m_{23}\xi_4 V + i\xi^\dagger{}_3\xi_5 m_{21}UV + i\xi^\dagger{}_3\xi_5 D_1\,V\,U + \\
& i\xi^\dagger{}_3\xi_5 D_3\,V\,W + i\xi^\dagger{}_3\rho\,D_3\xi_2 W + i\xi^\dagger{}_2\rho\,D_3\xi_1 W + \\
& i\xi^\dagger{}_2\frac{1}{\gamma M^2}\,\rho\,D_1\xi_3 + i\xi^\dagger{}_2\rho\,U D_1\xi_1 + i\xi^\dagger{}_2\frac{1}{\gamma M^2}\,D_1\rho\,\xi_3 + \\
& i\xi^\dagger{}_2\xi_5 D_3\,U\,W + i\xi^\dagger{}_2\rho\,m_{13}\xi_4 U + i\xi^\dagger{}_2\frac{1}{\gamma M^2}\,\xi_5 D_1\,T + \\
& i\xi^\dagger{}_2\rho\,D_1\,U\,\xi_1 + i\xi^\dagger{}_2\frac{1}{\gamma M^2}\,T D_1\xi_5 + i\xi^\dagger{}_2\xi_5 D_1\,U\,U - \\
& 2\,i\xi^\dagger{}_2\rho\,m_{21}V\xi_2 - i\xi^\dagger{}_2\xi_5 m_{21}V^2 + i\xi^\dagger{}_1 m_{13}\rho\,\xi_4 + \\
& i\xi^\dagger{}_1 m_{31}\rho\,\xi_1 + i\xi^\dagger{}_1 m_{31}\xi_5 U + i\xi^\dagger{}_1 m_{21}\rho\,\xi_1 + i\xi^\dagger{}_1 m_{23}\rho\,\xi_4 + \\
& i\xi^\dagger{}_1 U D_1\xi_5 + i\xi^\dagger{}_1\xi_5 D_3\,W + i\xi^\dagger{}_1\rho\,D_1\xi_1 + i\xi^\dagger{}_1 D_3\xi_5 W - \\
& i\xi^\dagger{}_5\xi_5\,cp\,W D_3\,T - i\xi^\dagger{}_5\rho\,\frac{1}{c_p}\frac{dc_p}{dT}\,\xi_3 U D_1\,T - \\
& i\xi^\dagger{}_5\rho\,cp\,W D_3\xi_3 - i\xi^\dagger{}_5\rho\,\frac{1}{c_p}\frac{dc_p}{dT}\,\xi_3 W D_3\,T - i\xi^\dagger{}_5\rho\,cp\,\xi_1 D_1\,T + \\
& i\xi^\dagger{}_5(\gamma-1)\,W\frac{1}{\gamma}\,D_3\rho\,\xi_3 + i\xi^\dagger{}_5(\gamma-1)\,W\frac{1}{\gamma}\,\rho\,D_3\xi_3 + \\
& i\xi^\dagger{}_5(\gamma-1)\,W\frac{1}{\gamma}\,\xi_5 D_3\,T + i\xi^\dagger{}_5\frac{\gamma-1}{\gamma}\,\gamma\,U\frac{1}{\gamma}\,D_1\rho\,\xi_3 + \\
& i\xi^\dagger{}_5(\gamma-1)\,U\frac{1}{\gamma}\,\rho\,D_1\xi_3 + i\xi^\dagger{}_5(\gamma-1)\,U\frac{1}{\gamma}\,T D_1\xi_5 + \\
& i\xi^\dagger{}_5(\gamma-1)\,M^2\,\xi_1 D_1 P + i\xi^\dagger{}_5(\gamma-1)\,U\frac{1}{\gamma}\,\xi_5 D_1\,T + \\
& i\xi^\dagger{}_5(\gamma-1)\,W\frac{1}{\gamma}\,D_3\xi_5 T - i\xi^\dagger{}_5\xi_5\,cp\,U D_1\,T - i\xi^\dagger{}_5\rho\,cp\,U D_1\xi_3 + \\
& i\xi^\dagger{}_4\rho\,D_3\xi_4 W + i\xi^\dagger{}_4\rho\,m_{31}U\xi_4 + i\xi^\dagger{}_4\rho\,U D_1\xi_4 + i\xi^\dagger{}_4\rho\,D_3\,W\,\xi_4 - \\
& 2\,i\xi^\dagger{}_4\rho\,m_{23}V\xi_2 - 2\,i\xi^\dagger{}_4\rho\,m_{13}U\xi_1 - i\xi^\dagger{}_4\xi_5 m_{23}V^2 - i\xi^\dagger{}_4\xi_5 m_{13}U^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\widehat{k}_2}{h_1^2} = \; & -i\xi^{\dagger}{}_2\rho\, D_{13}\, U\,\xi_4 - i\xi^{\dagger}{}_3\rho\, D_{13}\, V\,\xi_4 - i\xi^{\dagger}{}_4\frac{1}{\gamma M^2}\, D_1\rho\, D_3\xi_3 - \\
& i\xi^{\dagger}{}_3 D_1\rho\, D_3\, V\,\xi_4 - i\xi^{\dagger}{}_4\frac{1}{\gamma M^2}\, D_{13}\rho\,\xi_3 - i\xi^{\dagger}{}_4\frac{1}{\gamma M^2}\, D_3\xi_5 D_1\, T - \\
& i\xi^{\dagger}{}_4\frac{1}{\gamma M^2}\,\xi_5 D_{13}\, T + i\xi^{\dagger}{}_1\rho\, D_3\xi_4 m_{31} - i\xi^{\dagger}{}_2 D_1\rho\, D_3\, U\,\xi_4 - \\
& i\xi^{\dagger}{}_5(\gamma-1)\, M^2\,\xi_4 D_{13} P + i\xi^{\dagger}{}_1 D_3\rho\,\xi_4 m_{31} + \\
& i\xi^{\dagger}{}_4\frac{1}{\gamma M^2}\, D_3\xi_5 T m_{31} + i\xi^{\dagger}{}_4\frac{1}{\gamma M^2}\,\rho\, D_3\xi_3 m_{31} + \\
& i\xi^{\dagger}{}_4\frac{1}{\gamma M^2}\,\xi_5 D_3\, T\, m_{31} + i\xi^{\dagger}{}_4\frac{1}{\gamma M^2}\, D_3\rho\,\xi_3 m_{31} + \\
& i\xi^{\dagger}{}_5 D_1\rho\, cp\,\xi_4 D_3\, T + i\xi^{\dagger}{}_5\rho\, cp\,\xi_4 D_{13}\, T + i\xi^{\dagger}{}_2\rho\, D_3\, U\,\xi_4 m_{31} + \\
& i\xi^{\dagger}{}_3\rho\, D_3\, V\,\xi_4 m_{31} - i\xi^{\dagger}{}_1 D_{13}\rho\,\xi_4 - i\xi^{\dagger}{}_1 D_1\rho\, D_3\xi_4 - \\
& i\xi^{\dagger}{}_5\rho\, cp\,\xi_4 D_3\, T\, m_{31} + i\xi^{\dagger}{}_5(\gamma-1)\, M^2\,\xi_4 D_3 P\, m_{31} + \\
& i\xi^{\dagger}{}_5\rho\,\frac{1}{c_p}\frac{dc_p}{dT}\, D_1\, T\,\xi_4 D_3\, T + \xi^{\dagger}{}_2\alpha_0\frac{1}{\gamma M^2}\, D_1\rho\,\xi_3 - \\
& \xi^{\dagger}{}_2\alpha_0\frac{1}{\gamma M^2}\,\rho\,\xi_3 m_{11} + \xi^{\dagger}{}_2 D_1\rho\,\beta_0\xi_1 V + \xi^{\dagger}{}_2\alpha_0\frac{1}{\gamma M^2}\,\xi_5 D_1\, T - \\
& \xi^{\dagger}{}_1\rho\,\alpha_0\xi_1 m_{11} + \xi^{\dagger}{}_1 D_1\rho\,\alpha_0\xi_1 - \xi^{\dagger}{}_2\alpha_0\frac{1}{\gamma M^2}\,\xi_5 T m_{11} + \\
& \xi^{\dagger}{}_5(\gamma-1)\, M^2\, U\alpha_0\frac{1}{\gamma M^2}\,\xi_5 D_1\, T + \\
& \xi^{\dagger}{}_5(\gamma-1)\, M^2\, D_1\, U\,\alpha_0\frac{1}{\gamma M^2}\,\rho\,\xi_3 - \\
& \xi^{\dagger}{}_5(\gamma-1)\,\beta_0 V\frac{1}{\gamma}\,\xi_5 T m_{21} + \xi^{\dagger}{}_5(\gamma-1)\,\beta_0 D_1\, V\,\frac{1}{\gamma}\,\xi_5 T + \\
& \xi^{\dagger}{}_5(\gamma-1)\, D_1\, U\,\alpha_0\frac{1}{\gamma}\,\xi_5 T - \xi^{\dagger}{}_5(\gamma-1)\,\omega\,\frac{1}{\gamma}\,\xi_5 D_1\, T + \\
& \xi^{\dagger}{}_5(\gamma-1)\,\beta_0 V\frac{1}{\gamma}\, D_1\rho\,\xi_3 - \xi^{\dagger}{}_5\rho\,\frac{1}{c_p}\frac{dc_p}{dT}\, D_1\, T\, U\alpha_0\xi_3 + \\
& \xi^{\dagger}{}_5\rho\, cp\,\beta_0 V\xi_3 m_{21} + \xi^{\dagger}{}_5(\gamma-1)\, M^2\, U\alpha_0\frac{1}{\gamma M^2}\, D_1\rho\,\xi_3 + \\
& \xi^{\dagger}{}_5 D_1\rho\, cp\,\omega\,\xi_3 - \xi^{\dagger}{}_5(\gamma-1)\, M^2\,\omega\,\frac{1}{\gamma M^2}\, D_1\rho\,\xi_3 - \\
& \xi^{\dagger}{}_5\rho\, cp\, D_1\, U\,\alpha_0\xi_3 + \\
& \xi^{\dagger}{}_5\rho\, cp\, U\alpha_0\xi_3 m_{11} - \xi^{\dagger}{}_5\rho\,\frac{1}{c_p}\frac{dc_p}{dT}\, D_1\, T\,\beta_0 V\xi_3 - \xi^{\dagger}{}_1\beta_0\rho\,\xi_2 m_{21} - \\
& \xi^{\dagger}{}_1\beta_0\xi_5 V m_{21} - \xi^{\dagger}{}_5 D_1\rho\, cp\,\beta_0 V\xi_3 + \\
& \xi^{\dagger}{}_1\alpha_0\xi_5 D_1\, U + \xi^{\dagger}{}_1\beta_0 D_1\rho\,\xi_2 + \xi^{\dagger}{}_1\beta_0\xi_5 D_1\, V - \xi^{\dagger}{}_4 D_1\rho\,\omega\,\xi_4 - \\
& \xi^{\dagger}{}_3 D_1\rho\,\omega\,\xi_2 - \xi^{\dagger}{}_2 D_1\rho\,\omega\,\xi_1 - \xi^{\dagger}{}_1\alpha_0\xi_5 U m_{11} - \xi^{\dagger}{}_5 D_1\rho\, cp\, U\alpha_0\xi_3 + \\
& \xi^{\dagger}{}_5(\gamma-1)\, M^2\,\beta_0 V\frac{1}{\gamma M^2}\,\xi_5 D_1\, T - \xi^{\dagger}{}_5\rho\, cp\,\beta_0 D_1\, V\,\xi_3 + \\
& \xi^{\dagger}{}_5\rho\,\frac{1}{c_p}\frac{dc_p}{dT}\, D_1\, T\,\omega\,\xi_3 - \xi^{\dagger}{}_5(\gamma-1)\, U\alpha_0\frac{1}{\gamma}\,\xi_5 T m_{11} -
\end{aligned}
$$

$$
\begin{aligned}
&{\xi^\dagger}_5(\gamma - 1)\,\beta_0 V \frac{1}{\gamma}\,\rho\,\xi_3 m_{21} - {\xi^\dagger}_5(\gamma - 1)\,U \alpha_0 \frac{1}{\gamma}\,\rho\,\xi_3 m_{11} + \\
&{\xi^\dagger}_4 \rho\,\alpha_0 \xi_4 D_1\,U + {\xi^\dagger}_4 D_1\,\rho\,\alpha_0 \xi_4 U + {\xi^\dagger}_4 D_1\,\rho\,\beta_0 \xi_4 V + {\xi^\dagger}_4 \rho\,\beta_0 \xi_4 D_1\,V - \\
&{\xi^\dagger}_4 \rho\,\beta_0 \xi_4 V m_{21} + {\xi^\dagger}_3 \rho\,\alpha_0 \xi_2 D_1\,U + {\xi^\dagger}_3 \rho\,\beta_0 \xi_2 D_1\,V - \\
&{\xi^\dagger}_3 \beta_0 \frac{1}{\gamma M^2}\,\xi_5 T m_{21} + {\xi^\dagger}_3 D_1\,\rho\,\alpha_0 \xi_2 U + {\xi^\dagger}_3 D_1\,\rho\,\beta_0 \xi_2 V + \\
&{\xi^\dagger}_3 \beta_0 \frac{1}{\gamma M^2}\,D_1\,\rho\,\xi_3 + {\xi^\dagger}_3 \beta_0 \frac{1}{\gamma M^2}\,\xi_5 D_1\,T - {\xi^\dagger}_3 \beta_0 \frac{1}{\gamma M^2}\,\rho\,\xi_3 m_{21} - \\
&{\xi^\dagger}_3 \rho\,\beta_0 \xi_2 V m_{21} - {\xi^\dagger}_3 \rho\,\alpha_0 \xi_2 U m_{11} - {\xi^\dagger}_4 \rho\,\alpha_0 \xi_4 U m_{11} + \\
&{\xi^\dagger}_5(\gamma - 1)\,\beta_0 D_1\,V \frac{1}{\gamma}\,\rho\,\xi_3 - {\xi^\dagger}_2 \rho\,\beta_0 \xi_1 V m_{21} + {\xi^\dagger}_2 \rho\,\beta_0 \xi_1 D_1\,V + \\
&{\xi^\dagger}_2 \rho\,\alpha_0 \xi_1 D_1\,U - {\xi^\dagger}_2 \rho\,\alpha_0 \xi_1 U m_{11} + {\xi^\dagger}_2 D_1\,\rho\,\alpha_0 \xi_1 U
\end{aligned}
$$

Appendix E

Thermodynamic properties

The fluid is treated as a thermally ideal gas. Hence, the specific heat, the dynamic viscosity and the heat conductivity are a function of temperature, only. The following approximations are used, respectively:
1. Dynamic viscosity μ: (a) The two part Sutherland law

$$\mu(T) = 1.458 \cdot 10^{-6} \frac{T^{3/2}}{T + 110.4} [Ns/m^2] \qquad for \qquad T > 110.4$$

and

$$\mu(T) = 0.0693873 \cdot 10^{-6} T [Ns/m^2] \qquad for \qquad T \leq 110.4$$

(b)A 4^{th} degree polynomial fit to experimental data (see table 2)
2. Heat conductivity κ:
(a) Assuming constant Prandtl number and constant specific heat Sutherland´s law is used to calculate heat conductivity.
(b) Keye´s formula:

$$\kappa(T) = 2.648151 \cdot 10^{-3} T^{1/2} [1 + (245.4/T) 10^{-12/T}]^{-1} [W/mK] \text{ for } T > 80K$$

and

$$\kappa(T) = 9.335056752 \cdot 10^{-5} T [W/mK] \text{ for } T \leq 80K$$

(c) A 4^{th} degree polynomial fit to experimental data (see table 2)
3. Specific heat c_p:
(a) Constant specific heat using the value calculated from table 2 for reference temperature.
(b) A 4^{th} degree polynomial fit to experimental data (see table 2)

value=$a_0 + a_1 T + a_2 T^2 + a_3 T^3 + a_4 T^4$			
a_i	c_p $[J/kgK]$	μ $[Ns/m^2]$	κ $[W/mK]$
0	$1.058183878 \cdot 10^{3}$	$-1.561632014 \cdot 10^{-7}$	$-1.305884703 \cdot 10^{-3}$
1	$-4.52457049 \cdot 10^{-1}$	$7.957989891 \cdot 10^{-8}$	$1.099134492 \cdot 10^{-4}$
2	$1.141345435 \cdot 10^{-3}$	$-6.930149679 \cdot 10^{-11}$	$-6.84697087 \cdot 10^{-8}$
3	$-7.957390422 \cdot 10^{-7}$	$4.068157752 \cdot 10^{-14}$	$3.327083322 \cdot 10^{-11}$
4	$1.910858151 \cdot 10^{-10}$	$-9182486030 \cdot 10^{-18}$	$-5.397866355 \cdot 10^{-15}$

Table 2:Values of the coefficients used for the polynomial fit to the experimental data (from [1])

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Report title

Nonlocal instability analysis based on the multiple-scales method

Abstract

Multiple-scales technique (MSC) is used to examine the instability of non-parallel, compressible, quasi three-dimensional boundary layer flows. It models the kinematics and convective amplification of waves with weakly divergent or curved wave-rays and wave-fronts, propagating in a weakly non-uniform flow. The stability equations are put in a system of ordinary differential equations in a general orthogonal curvilinear coordinate system. The zeroth- order equations are homogeneous and govern the disturbance motion in a parallel flow and the non-local effects are calculated from the inhomogeneous first-order equations. The equations rewritten as a system of first order differential equations are discretized using compact finite difference scheme. For validation of the multiple-scales technique, we have compared the growth rates with results from 'parabolized stability equations' (PSE).

Keywords

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Sammanfattning

Multiple-scales (MSC) används för att undersöka instabilitet hos icke-parallella, kompressibla, kvasi tredimensionella gränsskikts flöden. Den modellerar kinematisk och konvektiv förstärkning av vågor med svagt divergenta eller krökta vågstrålar och vågfronter som propagerar i ett svagt icke-likformigt flöde. Stabilitets ekvationerna är skrivna i ett system av ordinära differential ekvationer i ett generellt ortogonalt krokformigt koordinat system. Nollte-ordningens ekvationer är homogena och bestämmer störningarnas rörelse i ett parallellt flöde och icke-lokala effekter är beräknade från de inhomogena första-ordningens ekvationer. Ekvationerna omskrivna som ett system av första ordningens differential ekvationer är diskretiserade med hjälp av ett kompakt finit differens schema. För validering av Multiple-scales tekniken, har vi jämfört tillväxthastighet med resultat från 'parabolized stability equations' (PSE).

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