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Scientific report

Sven-Lennart Wirkander

Particle Estimation, Basic Theory: an Introduction

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Report title

Particle Estimation, Basic Theory: an Introduction

Abstract

In the project "Jamming of Navigation Systems" different combinations of inertial and GPS-based navigation systems are investigated. A topic of special interest in this connection is the so called tightly coupled integration of the two types of systems. One reason for this is its importance as a prerequisite for the introduction of electrically steerable antennas, by means of which external disturbances can be efficiently suppressed, thereby increasing the robustness of the navigation system. However, the tightly coupled integration gives rise to nonlinear estimation problems. For this reason a Monte Carlo-based estimator called "Particle Filter" has been investigated in an earlier report and found to give a better performance in certain respects. It does not, as opposed to the Kalman estimator, require that the system noise is gaussian in order to work optimally.
This report consists of a derivation of the Particle Filter algoritm.

Keywords

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Rapportens titel

Introduktion till grundläggande teori för partikelestimering

Sammanfattning

I projektet ”Störning av navigeringssystem” studeras bl a kombinationer av tröghets- och GPS-baserade navigeringssystem. Härvid är s k tätt kopplad integration särskilt intressant, eftersom den bl a är en förutsättning för införande av s k elektriskt styrbara gruppantenner, med vars hjälp utifrån kommande störningar effektivt kan undertryckas, varigenom robustheten och störfastheten hos ett navigeringssystem ökar. Den tätt kopplade integrationen ger emellertid upphov till olinjära estimeringsproblem, varför en Monte Carlo-baserad s k partikelestimeringsalgoritm i en tidigare rapport har testats och jämförts med en konventionell Kalmanestimator och där befunnits ge bättre navigeringsprestanda i vissa avseenden. Den kräver inte, till skillnad från Kalmanestimatorn, att systembruset är gaussiskt för att fungera optimalt.
Föreliggande rapport består av en härledning partikelestimeringsalgoritmen.

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Contents

1. Introduction

1.1 Background

In the project "Jamming of Navigation System" different algorithms for achieving robustness against intentional jamming are investigated [4]. Here, in particular, an integrated GNSS (Global Navigation Satellite System)/INS (Inertial Navigation System) system in combination with an electrically steerable antenna array is analysed. The robustness of navigation systems is crucial and can be achieved in many ways. One way is to protect the GNSS receiver by using different software and hardware solutions. Another is to support the GNSS system with external complementary measurements. Two efficient methods have shown to be:

- Supporting the GNSS receiver with a complementary jamming robust sensor system, e.g. integrating GNSS and the IMU.
- Protecting the GNSS receiver by using smart antennas (adaptive beamforming antennas and switched beam antennas), thereby achieving spatial null steering or spatial beam forming.

Jamming of navigation systems is a current threat since GNSS became a key component in modern navigation systems. The use of the United States' Global Positioning System (GPS) is expanding, and recently the European Union has approved funding of the Galileo project, which is the European equivalent to GPS.

1.2 Aim of work

In order to construct navigation systems that are robust and insensitive to disturbances (intentional as well as untintentional), multisensor systems are often constructed, in which different sensors are combined together into redundant data fusion systems. In such a system, one of the main tasks is to calculate (estimate) in real time the position and/or velocity of a vehicle, based upon all these collected data. The problem of estimating the state of a dynamical system is also a central one in e g control theory building on feedback methods where system states are used to determine control signals. If the dynamical system that models reality is linear, or at least can be linearized with sufficient accuracy, and in addition has gaussian statistics, it is a well established fact that the Kalman estimation algorithm solves the optimization problem of minimizing the expected quadratic estimate error, [2], [5].

In those applications where the state dynamics are not linear, the method of linearizing the state dynamics around the current estimate and applying the Kalman algorithm ("Extended Kalman") has long been a common way to generalize as far as possible a method that has shown to be outstanding, given the restrictions of linearity and gaussianity. However, no general theoretical convergence proofs have been accomplished for the Extended Kalman method when these restrictions are violated, except possibly for certain special cases, and it is always uncertain whether or not it will work for all cases in a given application. This fact will, of course, be more stressed the more the system deviates from the linear/gaussian requirements.

A family of estimators, applicable to a much broader class of dynamical systems, has recently aroused interest for applications that have strong nonlinearities, such as hard limitations for the state values. Such limitations are often imposed in navigation applications if certain prior knowledge restricts the positions and/or velocities to some given subset of the state space.

These estimators use a great number of step-by-step simulations of a stochastic state model of the system, where the different simulated states ("particles") after each step are compared with measurements from the real system and assigned a quality number, used for weighing together the different particles to make up for the total state estimate.

Already in the sixties, similar Monte Carlo-based estimation methods were suggested. However, because of the extensive computational requirements, it was not practically useful at the time.

During the last years, a stream of papers have been published, starting with [1]. A recent application to navigation problems is found in [4]. The aim of this report is to give a mathematical derivation of the so called Particle Estimation algorithm that is based solely on Bayes' theorem and the system equations. This derivation is mainly a compilation and elucidation of the two descriptions in [3] and [6].

2. Basic relations

Consider the following discrete time dynamical system:

$$x_t = f(x_{t-1}) + w_t, \tag{2.1}$$
$$y_t = h(x_t) + e_t, \tag{2.2}$$

where the state trajectory is $\{x_t\}_{t=0}^{\infty} \subset R^n$, the process noise trajectory $\{w_t\}_{t=1}^{\infty} \subset R^n$, and the state transition function is $f : R^n \to R^n$. The measurement trajectory is $\{y_t\}_{t=0}^{\infty} \subset R^m$, the measurement noise trajectory $\{e_t\}_{t=0}^{\infty} \subset R^m$, and the measurement function is $h : R^n \to R^m$. The process and measurement noises are white sequences with probability density functions (pdf:s) $p_{w_t}(\cdot)$ and $p_{e_t}(\cdot)$, respectively.

We define the part of the state and measurement trajectories up to time t as

$$x_{0:t} \triangleq (x_0, x_1, ..., x_t), \tag{2.3}$$
$$y_{0:t} \triangleq (y_0, y_1, ..., y_t). \tag{2.4}$$

Eq.(2.1) and the whiteness of $\{w_t\}_{t=1}^{\infty}$ imply the Markov property of the state trajectory

$$p(x_t|x_{0:t-1}) = p(x_t|x_{t-1}) \tag{2.5}$$

(see Appendix 1), which in turn gives the following product expression for the state trajectory "prior" pdf $p(x_{0:t})$, i. e. the distribution of all the states up to time t without using any knowledge of the measurements:

$$\begin{aligned} p(x_{0:t}) &= p(x_{0:t-1})\, p(x_t|x_{0:t-1}) \\ \text{[eq. (2.5)]} &= p(x_{0:t-1})\, p(x_t|x_{t-1}) \\ &= ... \\ &= p(x_0) \prod_{k=1}^{t} p(x_k|x_{k-1}). \end{aligned} \tag{2.6}$$

Because, according to eq.(2.1), $p(x_t|x_{t-1}) = p_{w_t}(x_t - f(x_{t-1}))$, which is a known function, this product can also be written

$$p(x_{0:t}) = p(x_0) \prod_{k=1}^{t} p_{w_k}(x_k - f(x_{k-1})). \tag{2.7}$$

From eq.(2.2) and the whiteness of $\{e_t\}_{t=0}^{\infty}$ follows that the pdf for the measurement at time t is uniquely determined by the state at the same point in time. In particular, if the state trajectory up to time $t-1$, $x_{0:t-1}$, is given, then the pdf for the measurements up to the same point in time, $y_{0:t-1}$, does not depend on anything else, so that e. g. $p(y_{0:t-1}|y_t, x_{0:t}) = p(y_{0:t-1}|x_{0:t-1})$. By similar reason, $p(y_t|x_{0:t}) = p(y_t|x_t)$. For an example of a derivation of a similar formula, see Appendix 1. All this implies the following product formula for the conditional measurements:

$$\begin{aligned} p(y_{0:t}|x_{0:t}) &= p(y_{0:t-1}|y_t, x_{0:t})\, p(y_t|x_{0:t}) \\ \text{[eq. (2.2)]} &= p(y_{0:t-1}|x_{0:t-1})\, p(y_t|x_t) \end{aligned}$$

$$\begin{aligned} &= \ldots \\ &= \prod_{k=0}^{t} p\left(y_k | x_k\right). \end{aligned} \tag{2.8}$$

According to eq.(2.2), $p\left(y_t|x_t\right) = p_{e_t}\left(y_t - h\left(x_t\right)\right)$, so eq. (2.8) can be written

$$p\left(y_{0:t}|x_{0:t}\right) = \prod_{k=0}^{t} p_{e_k}\left(y_k - h\left(x_k\right)\right) \tag{2.9}$$

(compare with eq.(2.7) for the a priori state pdf).

For completeness we also state Bayes' theorem, applied to the state and measurement trajectories. This application of the theorem expresses the required distribution for the state trajectory at time t, given the observed measurements up to the same point in time, in terms of the "prior" state pdf and the conditional measurement trajectory pdf given the state trajectory, according to

$$p(x_{0:t}|y_{0:t}) = \frac{p(y_{0:t}|x_{0:t})p(x_{0:t})}{p(y_{0:t})}. \tag{2.10}$$

By means of eqn:s (2.2) and (2.5) we can derive a recursion for the desired pdf in eq.(2.10):

$$\begin{aligned} p(x_{0:t}|y_{0:t}) &= \\ [\text{eq. } (2.10)] &= \frac{p\left(y_t|y_{0:t-1}, x_{0:t}\right) p\left(y_{0:t-1}|x_{0:t}\right) p\left(x_t|x_{0:t-1}\right) p\left(x_{0:t-1}\right)}{p\left(y_t|y_{0:t-1}\right) p\left(y_{0:t-1}\right)} \\ [\text{eqn:s } (2.2)\,, (2.5)] &= \frac{p\left(y_t|x_t\right) p\left(y_{0:t-1}|x_{0:t-1}\right) p\left(x_t|x_{t-1}\right) p\left(x_{0:t-1}\right)}{p\left(y_t|y_{0:t-1}\right) p\left(y_{0:t-1}\right)} \\ &= \frac{p(y_t|x_t)p(x_t|x_{t-1})}{p\left(y_t|y_{0:t-1}\right)} \cdot \frac{p(y_{0:t-1}|x_{0:t-1})p(x_{0:t-1})}{p(y_{0:t-1})} \\ [\text{eq. } (2.10)] &= \frac{p(y_t|x_t)p(x_t|x_{t-1})}{p\left(y_t|y_{0:t-1}\right)} \cdot p\left(x_{0:t-1}|y_{0:t-1}\right). \end{aligned} \tag{2.11}$$

This formula is relevant for $t \geq 1$. For $t = 0$, Bayes can be directly applied to $p(x_0|y_0)$:

$$p(x_0|y_0) = \frac{p(y_0|x_0)p\left(x_0\right)}{p(y_0)}. \tag{2.12}$$

Observe that $p\left(y_t|x_t\right) = p_{e_t}\left(y_t - h\left(x_t\right)\right)$ for $t \geqslant 0$ and $p\left(x_t|x_{t-1}\right) = p_{w_t}\left(x_t - f\left(x_{t-1}\right)\right)$ for $t \geqslant 1$ and are therefore known functions, as is $p\left(x_0\right)$.

3. Expectation

Because calculating the whole pdf function $p(x_{0:t}|y_{0:t})$ from the recursion defined by eqn:s (2.11) and (2.12) is very cumbersome indeed, we may be satisfied with an estimation of the expected value of some function g_t, defined on $R^{n(t+1)}$, with regard to the desired conditional pdf $p(x_{0:t}|y_{0:t})$. The most common choice for g_t is the identity function, i. e. $g_t(x_{0:t}) = x_{0:t}$, in which case the expected value of g_t becomes the conditional mean of $x_{0:t}$, which in turn happens to be the minimum variance estimate of $x_{0:t}$, given the measurements $y_{0:t}$.

Now suppose also that $q(x_{0:t}|y_{0:t})$ is another conditional pdf, whose support contains the support of $p(x_{0:t}|y_{0:t})$ (for avoiding division by zero in the integrand in eq.(3.1) below). Then this expected value can be expressed both as a weighted integral of $g_t(x_{0:t})$ with weight $p(x_{0:t}|y_{0:t})$, and as a weighted integral of $g_t(x_{0:t})\frac{p(x_{0:t}|y_{0:t})}{q(x_{0:t}|y_{0:t})}$ with weight $q(x_{0:t}|y_{0:t})$, or

$$\begin{aligned} m^p &\triangleq E_{p(x_{0:t}|y_{0:t})}\left[g_t(x_{0:t})\right] \\ &= \int g_t(x_{0:t})\, p(x_{0:t}|y_{0:t}) dx_{0:t} \\ &= \int g_t(x_{0:t}) \frac{p(x_{0:t}|y_{0:t})}{q(x_{0:t}|y_{0:t})} q(x_{0:t}|y_{0:t}) dx_{0:t} \\ &= E_{q(x_{0:t}|y_{0:t})}\left[g_t(x_{0:t})\, w_t(x_{0:t})\right] \triangleq m^q, \end{aligned} \tag{3.1}$$

where

$$w_t(x_{0:t}) \triangleq \frac{p(x_{0:t}|y_{0:t})}{q(x_{0:t}|y_{0:t})}. \tag{3.2}$$

(The explicit t dependence in w_t is motivated by the $y_{0:t}$ in the p and q functions.) The covariance matrix of $g_t(x_{0:t})$ with respect to the distribution $p(x_{0:t}|y_{0:t})$ is

$$P^p \triangleq E_{p(x_{0:t}|y_{0:t})}\left[\left(g_t(x_{0:t}) - m^p\right)\left(g_t(x_{0:t}) - m^p\right)^T\right], \tag{3.3}$$

whereas the covariance of $g_t(x_{0:t})\, w_t(x_{0:t})$ with respect to $q(x_{0:t}|y_{0:t})$ is

$$P^q \triangleq E_{q(x_{0:t}|y_{0:t})}\left[\left(g_t(x_{0:t}) \frac{p(x_{0:t}|y_{0:t})}{q(x_{0:t}|y_{0:t})} - m^q\right)\left(g_t(x_{0:t}) \frac{p(x_{0:t}|y_{0:t})}{q(x_{0:t}|y_{0:t})} - m^q\right)^T\right]. \tag{3.4}$$

In general, $P^q \neq P^p$.

4. Weights recursion

The conditional pdf $q(x_{0:t}|y_{0:t})$ that was introduced in eq.(3.1) is called the "importance sampling distribution", and will be used later for the sampling of particles to the estimator. The quotient $w_t\left(x_{0:t}\right)$ between two conditional pdf:s as defined in eq.(3.2) is called the "importance weight", and will be used when the expectation in eq.(3.1) is approximated with the arithmetic mean of values sampled from the pdf $q(x_{0:t}|y_{0:t})$. In order for a recursion with respect to time for the $w_t\left(x_{0:t}\right)$ to be derived, we impose the following requirement on the q function:

$$q(x_{0:t-1}|y_{0:t}) = q(x_{0:t-1}|y_{0:t-1}). \tag{4.1}$$

This feature implies that, according to this pdf, no state will have a distribution that depends on "future" measurements. Therefore, an observation of a measurement never changes the probability distribution of "earlier" states. Such observations thus never cause the need for resampling of those earlier states in the particles (more about this later). So, by means of eqn:s (2.11), (4.1), and (3.2), we get a recursion for $w_t\left(x_{0:t}\right)$ in the following way:

$$\begin{aligned}
w_t\left(x_{0:t}\right)[\text{eq. (3.2)}] &= \frac{p(x_{0:t}|y_{0:t})}{q(x_{0:t}|y_{0:t})} \\
[\text{eq. (2.11)}] &= \frac{p(y_t|x_t)p(x_t|x_{t-1})p\left(x_{0:t-1}|y_{0:t-1}\right)}{p\left(y_t|y_{0:t-1}\right)q\left(x_t|x_{0:t-1},y_{0:t}\right)q\left(x_{0:t-1}|y_{0:t}\right)} \\
[\text{eq. (4.1)}] &= \frac{p(y_t|x_t)p(x_t|x_{t-1})}{p\left(y_t|y_{0:t-1}\right)q\left(x_t|x_{0:t-1},y_{0:t}\right)} \cdot \frac{p\left(x_{0:t-1}|y_{0:t-1}\right)}{q\left(x_{0:t-1}|y_{0:t-1}\right)} \\
[\text{eq. (3.2)}] &= \frac{p(y_t|x_t)p(x_t|x_{t-1})}{p\left(y_t|y_{0:t-1}\right)q\left(x_t|x_{0:t-1},y_{0:t}\right)} \cdot w_{t-1}\left(x_{0:t-1}\right).
\end{aligned} \tag{4.2}$$

This formula is relevant for $t \geq 1$. $t = 0$ gives the initial condition for the recursion:

$$w_0\left(x_0\right) = [\text{eq. (3.2)}] = \frac{p(x_0|y_0)}{q(x_0|y_0)} = [\text{eq. (2.12)}] = \frac{p(y_0|x_0)p(x_0)}{p(y_0)q(x_0|y_0)}. \tag{4.3}$$

5. Particle approximation I

The purpose now is to approximate the integral defined in eq.(3.1). The method described in this paper does this by means of stochastic sampling. Suppose that we can make one realization of the random variable (or, rather, the finite random process) $x_{0:t}$ according to the pdf $p(x_{0:t}|y_{0:t})$. The random variable transformed by the given function g_t , i.e. $g_t\left(x_{0:t}\right)$, then of course has the expected value $E_{p(x_{0:t}|y_{0:t})}\left[g_t\left(x_{0:t}\right)\right]$, i.e. the quantity we want to calculate according to eq.(3.1) . By this sampling from the pdf $p(x_{0:t}|y_{0:t})$, we can hope to get numerical values in the neighborhood of the true expectation, and the variances in the diagonal of P^p according to eq.(3.3) are measures of the expected errors. Of course, the smaller variances, the better estimation procedure. So, if we could create another random variable with the same expected value, but with smaller variances, and if we were able to sample from this new random variable, then we would have a better estimator. One random variable with these good features is the so called sample average, i.e. the arithmetic mean of a number of independent random variables that have the same distribution as the original one. Therefore, let $\left\{x_{0:t}^i\right\}_{i=1}^N$ be N independent, identically distributed (iid) random variables, or "particles", each one with pdf $p(x_{0:t}|y_{0:t})$, which means that $x_{0:t}^i$ has the same distribution as $x_{0:t}$, $\forall i$. Now form the new random variable

$$s_N^p \triangleq \frac{1}{N}\sum_{i=1}^{N} g_t\left(x_{0:t}^i\right), \tag{5.1}$$

i.e. the sample average of $\left\{g_t\left(x_{0:t}^i\right)\right\}_{i=1}^N$. This random variable has the same expected value with respect to $\prod_{j=1}^N p(x_{0:t}^j|y_{0:t})$ as $g_t\left(x_{0:t}\right)$has with respect to $p(x_{0:t}|y_{0:t})$, because

$$\begin{aligned}
m_N^p &\triangleq E_{\prod_{j=1}^N p(x_{0:t}^j|y_{0:t})}\left[s_N^p\right] \\
[\text{eq. (5.1)}] &= \frac{1}{N}\sum_{i=1}^{N} E_{p(x_{0:t}^i|y_{0:t})}\left[g_t\left(x_{0:t}^i\right)\right] \\
[\text{ident. distr.}] &= \frac{1}{N}\sum_{i=1}^{N} E_{p(x_{0:t}|y_{0:t})}\left[g_t\left(x_{0:t}\right)\right] \\
&= E_{p(x_{0:t}|y_{0:t})}\left[g_t\left(x_{0:t}\right)\right] \\
[\text{eq. (3.1)}] &= m^p.
\end{aligned} \tag{5.2}$$

The covariance matix of s_N^p with respect to $p(x_{0:t}|y_{0:t})$ is

$$\begin{aligned}
P_N^p &\triangleq E_{\prod_{j=1}^N p(x_{0:t}^j|y_{0:t})}\left[\left(s_N^p - m_N^p\right)\left(s_N^p - m_N^p\right)^T\right] \\
[\text{eq. (5.2)}] &= E_{\prod_{j=1}^N p(x_{0:t}^j|y_{0:t})}\left[\left(s_N^p - m^p\right)\left(s_N^p - m^p\right)^T\right] \\
[\text{eq. (5.1)}] &= \frac{1}{N^2} E_{\prod_{j=1}^N p(x_{0:t}^j|y_{0:t})}\left[\left(\sum_{i=1}^{N}\left(g\left(x_{0:t}^i\right) - m^p\right)\right)\left(\sum_{i=1}^{N}\left(g\left(x_{0:t}^i\right) - m^p\right)\right)^T\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E_{\prod_{j=1}^{N} p(x_{0:t}^{j}|y_{0:t})}\left[\left(g\left(x_{0:t}^{i}\right)-m^p\right)\left(g\left(x_{0:t}^{j}\right)-m^p\right)^T\right] \\
&= \frac{1}{N^2}\sum_{i=1}^{N} E_{p(x_{0:t}^{i}|y_{0:t})}\left[\left(g\left(x_{0:t}^{i}\right)-m^p\right)\left(g\left(x_{0:t}^{i}\right)-m^p\right)^T\right] \\
\text{[ident. distr.]} &= \frac{1}{N^2} N \cdot E_{p(x_{0:t}|y_{0:t})}\left[\left(g\left(x_{0:t}\right)-m^p\right)\left(g\left(x_{0:t}\right)-m^p\right)^T\right] \\
\text{[eq. (3.3)]} &= \frac{1}{N}P^p, \qquad (5.3)
\end{aligned}$$

i.e., it decreases inversely with the number of sampled random variables. Therefore s_N^p has the desired feature, so there seem to be good reasons to use it as an estimator of m^p. (In Appendix 2, it is is shown, by means of Chebyschev's inequality, that a decrease of the variance decreases "in probability" a sample's deviation from the expected value.)

However, though not theoretically impossible, the required sampling from $p(x_{0:t}|y_{0:t})$ is intractable to do in practice. A remedy of this is to choose another, known, distribution to sample from, which will be described next.

6. Particle approximation II

The quantity to be estimated, m^p, can according to eq.(3.1) be written either as $E_{p(x_{0:t}|y_{0:t})}[g_t(x_{0:t})]$, i.e. the expected value of $g_t(x_{0:t})$ with respect to the distribution $p(x_{0:t}|y_{0:t})$, or as $E_{q(x_{0:t}|y_{0:t})}[g_t(x_{0:t})\,w_t(x_{0:t})]$, i.e. the expected value of $g_t(x_{0:t})\,w_t(x_{0:t})$ with respect to the distribution $q(x_{0:t}|y_{0:t})$, which has the property of eq.(4.1), and where the random variable $w_t(x_{0:t})$ is defined in eq.(3.2).

Suppose now that the pdf $q(x_{0:t}|y_{0:t})$ is chosen such that it is possible to take independent samples from it. Create the sample average

$$s_N^q \triangleq \frac{1}{N}\sum_{i=1}^{N} g_t\left(x_{0:t}^i\right) w_t\left(x_{0:t}^i\right), \tag{6.1}$$

where $\left\{x_{0:t}^i\right\}_{i=1}^N$ now are iid random variables with pdf $q(x_{0:t}|y_{0:t})$. In eq.(3.1) it was shown that $m^q = E_{q(x_{0:t}|y_{0:t})}[g_t(x_{0:t})\,w_t(x_{0:t})] = E_{p(x_{0:t}|y_{0:t})}[g_t(x_{0:t})] = m^p$. The sample average in eq.(6.1) also has this same value as its expected value with respect to the pdf $\prod_{j=1}^N q(x_{0:t}^j|y_{0:t})$, because

$$\begin{aligned}
m_N^q &\triangleq E_{\prod_{j=1}^N q(x_{0:t}^j|y_{0:t})}\left[s_N^q\right] \\
[\text{eq. (6.1)}] \quad &= \frac{1}{N}\sum_{i=1}^{N} E_{\prod_{j=1}^N q(x_{0:t}^j|y_{0:t})}\left[g_t\left(x_{0:t}^i\right) w_t\left(x_{0:t}^i\right)\right] \\
[\text{ident. distr.}] \quad &= \frac{1}{N}\sum_{i=1}^{N}\int g_t(x_{0:t})\,w_t(x_{0:t})\,q(x_{0:t}|y_{0:t})dx_{0:t} \\
[\text{eq. (3.2)}] \quad &= \int g_t(x_{0:t})\,p(x_{0:t}|y_{0:t})dx_{0:t} \\
[\text{eq. (3.1)}] \quad &= m^p.
\end{aligned} \tag{6.2}$$

The covariance P_N^q for s_N^q decreases with N in the same way as the covariance P_N^p for s_N^p (see eq. (5.3)), although the fact that $P^q \neq P^p$ implies that $P_N^q \neq P_N^p$. Because of the condition imposed upon the importance sampling pdf in eq.(4.1), the sampling can be done recursively in time for each particle. Indeed, as pointed out earlier, this condition implies that a new measurement does not change the conditional distribution for the states at earlier points in time, i.e. $x_{0:t-1}$ is independent of y_t, which is exactly what eq.(4.1) tells us (notice that this is in general not the case with the pdf that we are looking for here, namely $p(x_{0:t}|y_{0:t})$, for which a new measurement, y_t, contains information about old states, $x_{0:t-1}$. This is the reason why the method described in section 5 cannot be used).

So, suppose that particle number i has been sampled up to and including time $t-1$ and has the realization x_{t-1}^i. The corresponding conditional likelihood value is $q(x_{t-1}^i|y_{0:t-1})$. Now draw a sample for the state at time t from the distribution with pdf $q(x_t|x_{0:t-1}^i, y_{0:t})$ and call it x_t^i. Because of the eq.(4.1) condition, the likelihood for the particle $x_{0:t}^i = \left(x_0^i, x_1^i, ..., x_{t-1}^i, x_t^i\right)$ will be $q\left(x_t^i|x_{0:t-1}^i, y_{0:t}\right) q\left(x_{0:t-1}^i|y_{0:t-1}\right) = q\left(x_{0:t}^i|y_{0:t}\right)$, which means that the total particle $x_{0:t}^i$ is drawn from the correct distribution. Once we have this realization of the particle $x_{0:t}^i$, we would like to be able to use the recursion formula eq.(4.2) (with initial condition eq.(4.3)) with $x^i_{0:t}$ substituted for $x_{0:t}$, in order to compute the corresponding importance weight $w_t\left(x^i_{0:t}\right)$ to be used in the estimate s^q_N of eq.(6.1).

However, in eq.(4.2) there is a difficulty with the quantity $p\left(y_t|y_{0:t-1}\right)$ in the denominator: although independent of the particle number i, it is unknown. It is true that the importance weights $\left\{w_t\left(x^i_{0:t}\right)\right\}^N_{i=1}$ can be calculated up to a proportionality constant, common to all particles, but this does not help us in calculating the sample estimate s^q_N. So we have to get rid of this factor.

7. Particle approximation III

In the previous section we tried to estimate the quantity $s^q = E_{q(x_{0:t}|y_{0:t})}\left[g_t(x_{0:t})\,w_t(x_{0:t})\right]$ by taking N samples from the distribution with pdf $q(x_{0:t}|y_{0:t})$ and calculate the arithmetic mean s_N^q according to eq.(6.1). This method failed because of problems with calculating the importance weights, $\left\{w_t\left(x_{0:t}^i\right)\right\}_{i=1}^N$. Therefore, after observing that $E_{q(x_{0:t}|y_{0:t})}\left[w_t(x_{0:t})\right] = \int w_t(x_{0:t})\,q(x_{0:t}|y_{0:t})\,dx_{0:t} = [eq.\,(3.2)] = \int p(x_{0:t}|y_{0:t})\,dx_{0:t} = 1$, we write the desired quantity as

$$s^q = \frac{E_{q(x_{0:t}|y_{0:t})}\left[g_t(x_{0:t})\,w_t(x_{0:t})\right]}{E_{q(x_{0:t}|y_{0:t})}\left[w_t(x_{0:t})\right]}, \tag{7.1}$$

and take samples $\left\{x_{0:t}^i\right\}_{i=1}^N$ from a distribution with pdf $q(x_{0:t}|y_{0:t})$ exactly as in the previous section.

Now, instead of taking the single sample average according to eq.(6.1), we form the quotient of the two sample averages corresponding to the nominator and the denominator of eq.(7.1), i.e.

$$\tilde{s}_N^q \triangleq \frac{\frac{1}{N}\sum_{i=1}^N g_t\left(x_{0:t}^i\right) w_t\left(x_{0:t}^i\right)}{\frac{1}{N}\sum_{i=1}^N w_t\left(x_{0:t}^i\right)} = \sum_{i=1}^N g_t\left(x_{0:t}^i\right)\tilde{w}_t\left(x_{0:t}^i\right), \tag{7.2}$$

where

$$\tilde{w}_t\left(x_{0:t}^i\right) \triangleq \frac{w_t\left(x_{0:t}^i\right)}{\sum_{j=1}^N w_t\left(x_{0:t}^j\right)}, \tag{7.3}$$

i.e. $\left\{\tilde{w}_t\left(x_{0:t}^i\right)\right\}_{i=1}^N$ are "normalized" importance weights with $\sum_{i=1}^N \tilde{w}_t\left(x_{0:t}^i\right) = 1$. In spite of the fact that $E_{\prod_{j=1}^N q\left(x_{0:t}^j|y_{0:t}\right)}\left[\frac{1}{N}\sum_{i=1}^N g_t\left(x_{0:t}^i\right) w_t\left(x_{0:t}^i\right)\right] = [\text{eq.}\,(6.2)] = m^p = [\text{eq.}\,(3.1)] = m^q$ and $E_{\prod_{j=1}^N q\left(x_{0:t}^j|y_{0:t}\right)}\left[\frac{1}{N}\sum_{i=1}^N w_t\left(x_{0:t}^i\right)\right] = \begin{bmatrix}\text{ident.}\\ \text{distr.}\end{bmatrix} = \frac{1}{N}\sum_{i=1}^N \int p(x_{0:t}|y_{0:t})\,dx_{0:t} = 1$, the expected value of the quotient,i.e. $E_{\prod_{j=1}^N q\left(x_{0:t}^j|y_{0:t}\right)}\left[\tilde{s}_N^q\right]$ is in general $\neq m^q$. However, $\lim p_{N\to\infty}\tilde{s}_N^q = m^q$ according to the strong law of large numbers. So eq. (7.2) can be used in about the same way as the one we tried with s_N^q. The recursion of eq. (4.2) with $x_{0:t} = x_{0:t}^i$ can now be replaced by

$$w_t\left(x_{0:t}^i\right) = \frac{p(y_t|x_t^i)p(x_t^i|x_{t-1}^i)}{q\left(x_t^i|x_{0:t-1}^i, y_{0:t}\right)} \cdot \tilde{w}_{t-1}\left(x_{0:t-1}^i\right) \tag{7.4}$$

for $i = 1, ..., N$, followed by

$$\tilde{w}_t\left(x_{0:t}^i\right) \triangleq \frac{w_t\left(x_{0:t}^i\right)}{\sum_{j=1}^N w_t\left(x_{0:t}^j\right)}. \tag{7.5}$$

for $i = 1, ..., N$. Notice that comparing with eq.(4.2), the factor $p(y_t|y_{o:t-1})$ in the denominator of eq. (7.4) is lacking. This is possible because the normalization in eq. (7.5) cancels all factors that are independent of the particle number. Therefore, the problem noticed at the end of section 5 has disappeared, and we have now a useful recursion procedure for estimationg m^p.

8. Choice of importance sampling distribution

In the treatment above, the function $q\left(x_{0:t}|y_{0:t}\right)$ has been quite general. The requirements that have been posed on it hitherto are

- it has to be a (conditional) pdf, i.e. $\int q\left(x_{0:t}|y_{0:t}\right) dx_{0:t} = 1$,
- $q\left(x_{0:t}|y_{0:t}\right) \neq 0$ for all $x_{0:t}$ where $p\left(x_{0:t}|y_{0:t}\right) \neq 0$ (because of eq. (3.1)),
- $q(x_{0:t-1}|y_{0:t}) = q(x_{0:t-1}|y_{0:t-1})$ (see eq. (4.1)), and
- it shall be possible to take independent samples from the pdf $q\left(x_{0:t}|y_{0:t}\right)$.

Within these constraints, $q\left(x_{0:t}|y_{0:t}\right)$ can be freely chosen.

One distribution that satisfies these requirements (except possibly the second) is the "prior" for the state with the pdf $p\left(x_{0:t}\right)$, i.e. the state a priori distribution that we would have without any observation of measurements. An explicit expression of this function is given by eq.(2.7), where the initial state pdf $p\left(x_0\right)$ is assumed to be known, and $p_{w_t}\left(\cdot\right)$ is a known function. The third requirement, namely that a measurement should not change the distribution for the old states, is ceartainly fulfilled, as $p\left(x_{0:t}\right)$ does not depend on the measurements at all.

The method for sampling of particles, as required in the last point above, is an application of the description in section 6, and will be as follows: suppose that the trajectory $x^i_{0:t-1}$ for particle number i, up to and including time $t-1$, is known. The conditional pdf for its state at time t, x^i_t, is then $p_{w_t}\left(x_t - f\left(x^i_{t-1}\right)\right)$, according to eq.(2.1) where, as we know, $p_{w_t}\left(\cdot\right)$ is a known pdf. So we just sample a value from this pdf and calculate our value for x^i_t by adding $f\left(x^i_{t-1}\right)$. That the total trajectory up to and including time t, $x^i_{0:t}$, is in this way indeed drawn from the pdf $p\left(x_{0:t}\right)$, can be seen from the fact that $p_{w_t}\left(x_t - f\left(x_{t-1}\right)\right) p\left(x_{0:t-1}\right) = [\text{eq. (2.1)}] = p\left(x_t|x_{t-1}\right) p\left(x_{0:t-1}\right) = [\text{eq. (2.5)}] = p\left(x_t|x_{0:t-1}\right) p\left(x_{0:t-1}\right) = p\left(x_{0:t}\right)$. Eq. (7.4) will be considerably simplified by this choice of importance sampling distribution, because of the Markov property of eq. (2.5). According to this property, $q\left(x^i_t|x^i_{0:t-1}, y_{0:t}\right) = p\left(x^i_t|x^i_{0:t-1}\right) = p\left(x^i_t|x^i_{t-1}\right)$, which reduces eq. (7.4) to

$$w_t\left(x^i_{0:t}\right) = p(y_t|x^i_t) \cdot \tilde{w}_{t-1}\left(x^i_{0:t-1}\right). \tag{8.1}$$

This means that the particle weights are only affected by the received measurement and not of the system dynamics or the state history for the particle in question.

9. Appendix 1

9.1 Derivation of Markov property for $\{x_t\}_0^\infty$

System equation: $x_k = f(x_{k-1}) + w_k$, where $\{w_k\}_0^\infty$ is a white process independent of x_0.

$$p(x_t|x_{0:t-1}) = \frac{p(x_t, x_{0:t-1})}{p(x_{0:t-1})} = \frac{p(x_0, x_1, \ldots, x_{t-1}, x_t)}{p(x_0, x_1, \ldots, x_{t-1})} = \frac{p(x_0, w_1, w_2, \ldots, w_{t-1}, w_t)}{p(x_0, w_1, w_2, \ldots, w_{t-1})}$$
$$= \frac{p(x_0)\prod_{k=1}^{t} p_{w_k}(x_k - f(x_{k-1}))}{p(x_0)\prod_{k=1}^{t-1} p_{w_k}(x_k - f(x_{k-1}))} = p_{w_t}(x_t - f(x_{t-1})) = p(x_t|x_{t-1})$$

9.2 Derivation of features for $\{y_t\}_0^\infty$ used on page 4

System equation: $y_k = h(x_k) + e_k$, where $\{e_k\}_0^\infty$ is a white process.

$$p(y_t|y_{0:t-1}, x_{0:t}) = \frac{p(y_t, y_{0:t-1}, x_{0:t})}{p(y_{0:t-1}, x_{0:t})} = \frac{p(y_0, y_1, \ldots, y_{t-1}, y_t, x_0, \ldots, x_t)}{p(y_0, y_1, \ldots, y_{t-1}, x_0, \ldots, x_t)}$$
$$= \frac{p(e_0, e_1, \ldots, e_{t-1}, e_t)}{p(e_0, e_1, \ldots, e_{t-1}, x_t)} = \frac{\prod_{k=0}^{t} p_{e_k}(y_k - h(x_{k-1}))}{\prod_{k=0}^{t-1} p_{e_k}(y_k - h(x_k)) \cdot p(x_t)} = \frac{p_{e_t}(y_t - h(x_{t-1}))}{p(x_t)}$$
$$= p(y_t - h(x_t)|x_t) = p(y_t|x_t)$$

10. Appendix 2

10.1 Chebychev's inequality for probabilities

Suppose X is a random variable with pdf $p(\cdot)$, $E[X] = m_X$
and $E\left[(X - m_X)^2\right] = \sigma_X^2$.

Then

$$\sigma_X^2 = \int_R (x - m_X)^2 p(x)\,dx \geq \int_{|x-m_X|>\varepsilon} (x - m_X)^2 p(x)\,dx$$

$$\geq \varepsilon^2 \int_{|x-m_X|>\varepsilon} p(x)\,dx = \varepsilon^2 P(|X - m_X| > \varepsilon)$$

for arbitrary $\varepsilon > 0$.

Thus

$$P(|X - m_X| > \varepsilon) \leq \frac{\sigma_X^2}{\varepsilon^2} \quad \text{(useful only for small } \frac{\sigma_X^2}{\varepsilon^2} \text{ !)}$$

Now let $\{X_i\}_1^n$ be n iid r.v.:s, each with pdf $p(\cdot)$.

Define $Y_n = \frac{1}{n}\sum_1^n X_i$. Then $E[Y_n] = m_X$ and $E\left[(Y_n - m_X)^2\right] = \frac{1}{n}\sigma_X^2$.

Apply Chebychev's inequality to Y_n :

$$P(|Y_n - m_X| > \varepsilon) \leq \frac{1}{n}\frac{\sigma_X^2}{\varepsilon^2}.$$

This motivates sampling from Y_n instead of X.

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