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Conservative Difference Formulations, Energy Estimates and Artificial Dissipation

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Abstract

An artificial dissipation term for linear and nonlinear hyperbolic Cauchy problems is determined such that we obtain an energy estimate despite a conservative formulation of the problems. The differential equations are solved using second, fourth and sixth order accurate difference operators, which all satisfy summation-by-parts properties. The dissipation terms are computed such that there is no loss of accuracy.

Contents

1 Introduction

Conservative difference methods for solving linear and nonlinear hyperbolic problems are often used. Mainly because the shock speed is calculated correctly [1]. However, to obtain an energy estimate for the Cauchy problems with a conservative method requires artificial dissipation. The energy estimate is necessary for obtaining strict stability.

Normally, the artificial dissipation is constructed in order to absorb the energy of unresolved modes in the problem. It can also be added to enable the calculation of problems involving shocks [6]. In this paper we aim for a particular kind of artificial dissipation that makes it possible to obtain an energy estimate despite a basic conservative difference approximation.

The artificial dissipation is constructed by expressing the conservative formulation as a skew-symmetric formulation with an artificial dissipation term added to it. The size and form of the artificial dissipation term depends on the specific problem to solve, the size of the mesh and the order of accuracy of the difference operators we use. Second, fourth and sixth order of accurate summation-by-parts (SBP) operators [2] with diagonal norms are used.

2 The linear problem

2.1 The continuous case

Consider the linear system of equations

$$\begin{aligned} u_t + (a(x)u)_x &= 0 \\ v_t + (b(x)v)_x &= 0 \end{aligned} \tag{1}$$

where $a(x) > 0$, $b(x) < 0$, $x \in [0 \;\; 1]$ and $t > 0$. The boundary conditions are determined by

$$u(0,t) = \alpha v(0,t), \qquad v(1,t) = \beta u(1,t) \tag{2}$$

and the initial conditions are

$$u(x,0) = f(x), \qquad v(x,0) = g(x). \tag{3}$$

The constants α and β in (2) will be determined later.

Multiplication of equation (1) with u, v and integration over the domain leads to

$$\frac{d}{dt}(\|u\|^2_{\ell_2} + \|v\|^2_{\ell_2}) = -(au^2 + bv^2)|_0^1 - \int_0^1 (a_x u^2 + b_x v^2)dx. \tag{4}$$

In (4) we have introduced the norm $\|u\|^2_{\ell_2} = \int_0^1 u^2 dx$. By choosing

$$\alpha = \sqrt{-\frac{b(0)}{a(0)}} \qquad \beta = \sqrt{-\frac{a(1)}{b(1)}}$$

in (2), we eliminate the boundary terms in (4) such that the final expression for the energy rate becomes

$$\frac{d}{dt}(\|u\|^2_{\ell_2} + \|v\|^2_{\ell_2}) = -\int_0^1 (a_x u^2 + b_x v^2)dx. \tag{5}$$

2.2 The discrete case

2.2.1 Stability

To prevent a discrete solution from contamination of unresolved features in long time integrations, we aim for a strictly stable method [3]. That concept is defined below.

Consider the following initial-boundary problem.

$$\begin{aligned} u_t + H\Big(x, t, \tfrac{\partial}{\partial x}\Big)u &= F, & x \in \Omega, \quad t \geq 0 \\ u &= f(x) & x \in \Omega, \quad t = 0 \\ u &= g(t) & x \in \Gamma, \quad t \geq 0 \end{aligned} \tag{6}$$

where $\Omega = \{x; 0 \leq x \leq 1\}$ and $\Gamma = \{x; x = 0\}$.

Definition 1. *(6) is said to be strongly well posed if an unique solution exists and the estimate*

$$\begin{aligned} &||u(\cdot,t)||_\Omega^2 + \textstyle\int_0^t ||u(\cdot,\tau)||_\Gamma^2 d\tau \\ &\leq K_c e^{\eta_c t}\Big(||u(\cdot,0)||_\Omega^2 + \textstyle\int_0^t (||F(\tau)||_\Omega^2 + ||g(\tau)||_\Gamma^2) d\tau\Big) \end{aligned} \tag{7}$$

holds. K_c and η_c do not depend on F, f or g. $||\cdot||_\Gamma$ and $||\cdot||_\Omega$ are suitable continuous norms.

The corresponding semi-discrete problem is

$$\begin{aligned} \mathbf{u}_t + H\Big(x_j, t, \tfrac{\partial}{\partial x_j}\Big)\mathbf{u} &= \mathbf{F}, & x_j \in \Omega, \quad t \geq 0 \\ \mathbf{u} &= \mathbf{f} & x_j \in \Omega, \quad t = 0 \\ \mathbf{u} &= \mathbf{g}(t) & x_j \in \Gamma, \quad t \geq 0 \end{aligned} \tag{8}$$

Definition 2. *(8) is said to be strongly stable if, for a sufficiently small Δx, there is an unique solution that satisfies*

$$\begin{aligned} &||\mathbf{u}||_\Omega^2 + \textstyle\int_0^t ||\mathbf{u}||_\Gamma^2 d\tau \\ &\leq K_d e^{\eta_d t}\Big(||\mathbf{u}||_\Omega^2 + \textstyle\int_0^t (||\mathbf{F}||_\Omega^2 + ||\mathbf{g}||_\Gamma^2) d\tau\Big) \end{aligned} \tag{9}$$

K_d and η_d do not depend on $\mathbf{F}, \mathbf{f}$ or $\mathbf{g}$. $||\cdot||_\Gamma$ and $||\cdot||_\Omega$ are suitable discrete norms.

Definition 3. *We call (8) strictly stable if the growth rates in (7) and (9) satisfy*

$$\eta_d \leq \eta_c + O(\Delta x) \tag{10}$$

2.2.2 Conservative method

A semi-discrete representation of (1), (2) and (3) is

$$\begin{array}{rcl} \mathbf{u}_t + D(A\mathbf{u}) & = & 0 \\ \mathbf{v}_t + D(B\mathbf{v}) & = & 0 \\ \mathbf{u}(0) & = & \mathbf{f} \\ \mathbf{v}(0) & = & \mathbf{g} \\ u_0(t) & = & \alpha v_0(t) \\ v_N(t) & = & \beta u_N(t) \end{array} \quad (11)$$

where A and B are diagonal matrices

$$A = \begin{pmatrix} a_0 & 0 & \dots \\ 0 & a_1 & \\ \vdots & & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} b_0 & 0 & \dots \\ 0 & b_1 & \\ \vdots & & \ddots \end{pmatrix},$$

with the values of a and b injected on the diagonal. The vectors $\mathbf{u}$ and $\mathbf{v}$ are the discrete representations of u and v. The system (11) is obviously on conservation form [3]. D is a spatial difference operator of the summation-by-parts (SBP) type [2]. A SBP operator D can be written as a product between two matrices, $P^{-1}Q$ that satisfy the following properties:

1. The matrix P is symmetric and positive definite, and $\Delta x p I \leq P \leq \Delta x q I$, where $p > 0$ and $q > 0$, both independent of the number of node points, $N+1$.

2. The matrix Q is nearly skew symmetric, ie
$Q + Q^T = diag(-1\ 0 \dots 0\ 1) = B$.

In addition, P must be a diagonal matrix in our case since we will later require that $AP = PA$ and $BP = PB$. Let $\mathbf{w} = (\mathbf{u}, \mathbf{v})^T$,

$$\mathcal{P} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}, \mathcal{Q} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}, \mathcal{F} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad (12)$$

and $\mathcal{B} = \mathcal{Q} + \mathcal{Q}^T$. Using the Simultaneuos approximation term (SAT) method [4], which makes the boundary conditions part of the difference equation through a "penalty"-term, and the definitions in (12) we can write (11) as

$$\mathbf{w}_t + \mathcal{D}\mathcal{F}\mathbf{w} = \mathcal{P}^{-1}S\mathbf{w} \quad (13)$$

where $\mathcal{D} = \mathcal{P}^{-1}\mathcal{Q}$ and S is a $(2N+2) * (2N+2)$ matrix with nonzero elements at position 1 and $N+2$ in the first row and $N+1$ and $2N+2$ in the last row.

$$S = \begin{pmatrix} \sigma_L & 0 & \dots & -\sigma_L\alpha & 0 & \dots \\ 0 & 0 & & & & \\ \vdots & & \ddots & & & \\ & & & & & \vdots \\ & & & & 0 & 0 \\ & & -\sigma_R\beta & 0 & \dots & \sigma_R \end{pmatrix}$$

Multiplying equation (13) with $\mathbf{w}^T\mathcal{P}$ from the left and adding the transposed resulting equation we get

$$\mathbf{w}^T\mathcal{P}\mathbf{w}_t + \mathbf{w}_t^T\mathcal{P}^T\mathbf{w} = \mathbf{w}^T(S+S^T)\mathbf{w} - \mathbf{w}^T(\mathcal{Q}\mathcal{F} + (\mathcal{Q}\mathcal{F})^T)\mathbf{w}$$

By using $\mathcal{Q} + \mathcal{Q}^T = \mathcal{B}$ we get

$$\begin{aligned} \tfrac{d}{dt}\|\mathbf{w}\|_{\mathcal{P}}^2 &= \mathbf{w}^T(S+S^T)\mathbf{w} - \mathbf{w}^T((\mathcal{B}-\mathcal{Q}^T)\mathcal{F} + \mathcal{F}\mathcal{Q}^T)\mathbf{w} \\ &= BT_1 + (\mathcal{D}\mathbf{w}, \mathcal{F}\mathbf{w})_{\mathcal{P}} - (\mathcal{D}\mathcal{F}\mathbf{w}, \mathbf{w})_{\mathcal{P}} \end{aligned}$$

where $BT_1 = \mathbf{w}^T(S + S^T - \mathcal{B}\mathcal{F})\mathbf{w}$ and $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T P\mathbf{v}$. Let

$$GR1 = (\mathcal{D}\mathbf{w}, \mathcal{F}\mathbf{w})_{\mathcal{P}} - (\mathcal{D}\mathcal{F}\mathbf{w}, \mathbf{w})_{\mathcal{P}}. \tag{14}$$

To obtain strict stability, a first requirement is that BT_1 has to be less than or equal to zero, that is $(S+S^T-\mathcal{B}\mathcal{F})$ has to be negative semi definite. By choosing σ_L and σ_R to be

$$\sigma_L = -a_0, \qquad \sigma_R = -b_N, \tag{15}$$

the eigenvalues of $(S + S^T - \mathcal{B}\mathcal{F})$ are non-positive.

2.2.3 Skew-symmetric method

Another formulation of (1) is

$$\begin{aligned} u_t + \frac{1}{2}a(x)_x u + \frac{1}{2}((a(x)u)_x + a(x)u_x) &= 0 \\ v_t + \frac{1}{2}b(x)_x v + \frac{1}{2}((b(x)v)_x + b(x)v_x) &= 0, \end{aligned} \tag{16}$$

where we have used the relation $(au)_x = \gamma(au)_x + (1-\gamma)(au_x + a_x u)$. In this case we choose $\gamma = \frac{1}{2}$. The semi discrete correspondence to (16) has the following appearence

$$\begin{aligned} \mathbf{u}_t + \tfrac{1}{2}A_x\mathbf{u} + \tfrac{1}{2}(DA\mathbf{u} + AD\mathbf{u}) &= 0 \\ \mathbf{v}_t + \tfrac{1}{2}B_x\mathbf{v} + \tfrac{1}{2}(DB\mathbf{v} + BD\mathbf{v}) &= 0 \end{aligned}$$

which is equivalent to

$$\mathbf{w}_t + \frac{1}{2}(\mathcal{F}_x\mathbf{w} + \mathcal{D}\mathcal{F}\mathbf{w} + \mathcal{F}\mathcal{D}\mathbf{w}) = 0 \tag{17}$$

where $\mathcal{F}_x = diag(A_x\ B_x)$ and

$$A_x = \begin{pmatrix} a_{x0} & & \\ & \ddots & \\ & & a_{xN} \end{pmatrix}, B_x = \begin{pmatrix} b_{x0} & & \\ & \ddots & \\ & & b_{xN} \end{pmatrix}.$$

Equation (17) augmented with the SAT term becomes

$$\mathbf{w}_t + \frac{1}{2}\mathcal{F}_x\mathbf{w} + \frac{1}{2}(\mathcal{D}\mathcal{F}\mathbf{w} + \mathcal{F}\mathcal{D}\mathbf{w}) = \mathcal{P}^{-1}S\mathbf{w}, \tag{18}$$

where the S-matrix is the same as in section 2.2.2. By multiplying (18) with $\mathbf{w}^T\mathcal{P}$ from the left and then adding the corresponding transposed equation, we get

$$\begin{aligned}\mathbf{w}^T\mathcal{P}\mathbf{w}_t + \mathbf{w}_t^T\mathcal{P}^T\mathbf{w} &= \mathbf{w}^T(S+S^T)\mathbf{w} - \tfrac{1}{2}\mathbf{w}^T(\mathcal{P}\mathcal{F}_x + (\mathcal{P}\mathcal{F}_x)^T)\mathbf{w} - \\ &\quad \tfrac{1}{2}\mathbf{w}^T(\mathcal{Q}\mathcal{F} + \mathcal{F}^T\mathcal{Q}^T + \mathcal{P}\mathcal{F}\mathcal{D} + \mathcal{P}\mathcal{F}\mathcal{D}^T)\mathbf{w}\end{aligned}$$

Now since $\mathcal{P}$ and $\mathcal{F}$ commute we get $\mathcal{P}\mathcal{F}\mathcal{D} = \mathcal{P}\mathcal{F}\mathcal{P}^{-1}\mathcal{Q} = \mathcal{F}\mathcal{Q}$ which leads to

$$\begin{aligned}\frac{d}{dt}\|\mathbf{w}\|^2_{\mathcal{P}} &= \mathbf{w}^T(S+S^T-\mathcal{B}\mathcal{F})\mathbf{w} - (\mathcal{F}_x\mathbf{w})^T\mathcal{P}\mathbf{w} - \\ &\quad \tfrac{1}{2}\mathbf{w}^T(-\mathcal{Q}^T\mathcal{F} + \mathcal{F}^T\mathcal{Q}^T - \mathcal{F}\mathcal{Q}^T + \mathcal{Q}^T\mathcal{F}^T)\mathbf{w} \\ &= BT_2 - (\mathcal{F}_x\mathbf{w}, \mathbf{w})_{\mathcal{P}}\end{aligned} \tag{19}$$

The boundary term $BT_2 = \mathbf{w}^T(S+S^T-\mathcal{B}\mathcal{F})\mathbf{w}$ equals BT_1 in section 2.2.2 and is therefore negative semi definite if (15) is used. Let

$$GR2 = -(\mathcal{F}_x\mathbf{w}, \mathbf{w})_{\mathcal{P}}. \tag{20}$$

2.2.4 Energy estimate

To obtain an energy estimate where the growth rate corresponds to the continuous case, $GR1$ in (14) for the conservative formulation and $GR2$ in (20) for skew-symmetric formulation must correspond to $-\int_0^1(a_xu^2 + b_xv^2)dx$ in (5). Otherwise, strict stability will not be obtained.

Using a second order SBP operator, $a(x) = 1 + \epsilon x$ and $b(x) = -1 + \epsilon x$ imply that

$$GR2 = -(\mathcal{F}_x\mathbf{w}, \mathbf{w})_{\mathcal{P}} = -\epsilon\|\mathbf{w}\|^2_{\mathcal{P}}.$$

This means that the discrete and continuous energy estimate resemble each other, i.e we get

$$\frac{d}{dt}\|\mathbf{w}\|^2_{\mathcal{P}} = -\epsilon\|\mathbf{w}\|^2_{\mathcal{P}}, \tag{21}$$

which mimics (5) perfectly and therefore a correct discrete spectrum is obtained, see figures 1 and 2 in section 2.2.5. However, since

$$GR1 = -\epsilon\|\mathbf{w}\|^2_{\mathcal{P}} + E,$$

the difference between $GR1$ and the correct discrete energy rate can be written

$$E = \epsilon\|\mathbf{w}\|^2_{\mathcal{P}} + GR1 = \frac{\epsilon\Delta x}{2}\sum_{i=1}^{N}((u_{i-1}-u_i)^2 + (v_{i-1}-v_i)^2).$$

The deviation E does not necessarily vanish with decreasing Δx, [5]. Consequently, the spectrum might not be correct, see figures 3 and 4 in section 2.2.5.

Note that if $\mathbf{u}$ and $\mathbf{v}$ in (20) are smooth functions, then

$$GR2 \approx -\int_0^1 (a_x u^2 + b_x v^2) dx.$$

This implies that the energy rate (19) for the skew-symmetric method correspond to (5). Note also that (20) can always be estimated as $GR2 \leq |\mathcal{F}_x|_{max} ||\mathbf{w}||^2_{\mathcal{P}}$ which leads to an energy estimate even though strict stability cannot be obtained.

2.2.5 Continuous- and discrete spectrum

By Laplace transforming (1) we get

$$\begin{aligned} s\tilde{u} - f(x) + (a\tilde{u})_x &= 0 \\ s\tilde{v} - g(x) + (b\tilde{v})_x &= 0, \end{aligned} \tag{22}$$

where

$$u(s) = \int_0^\infty u e^{-st} dt.$$

The solutions to (22) with $f(x) = g(x) = 0$, are

$$\tilde{u} = \frac{C_1}{|a| \int_{\delta_1}^x \frac{s}{a} dx}, \quad \tilde{v} = \frac{C_2}{|b| \int_{\delta_2}^x \frac{s}{b} dx}. \tag{23}$$

C_1 and C_2 are constants and δ_1 and δ_2 are arbitrary real numbers. The solutions $\tilde{u}$ and $\tilde{v}$ in (23) inserted in the boundary conditions (2) leads to

$$M \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \tilde{u}_1(0,s) & -\alpha\tilde{v}_1(0,s) \\ \beta\tilde{u}_1(1,s) & -\tilde{v}_1(1,s) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0,$$

where $\tilde{u}_1 = \tilde{u}/C_1$ and $\tilde{v}_1 = \tilde{v}/C_2$. The spectrum is determined by solving

$$|M| = 0$$

for the s values. For general $a(x)$ and $b(x)$ the spectrum is given by

$$s = \frac{ln(\alpha\beta) + 2n\pi i}{\int_1^0 a(x)^{-1} dx + \int_0^1 b(x)^{-1} dx},$$

where i is the imaginary unit and $n \in \mathbb{Z}$. The discrete spectrum is given by computing the eigenvalues of G

$$\mathbf{w}_t = G\mathbf{w} \tag{24}$$

Figure 1. Skew-symmetric spectrum of G, second order case. $a = 1+0.8x, b = -1+0.8x$

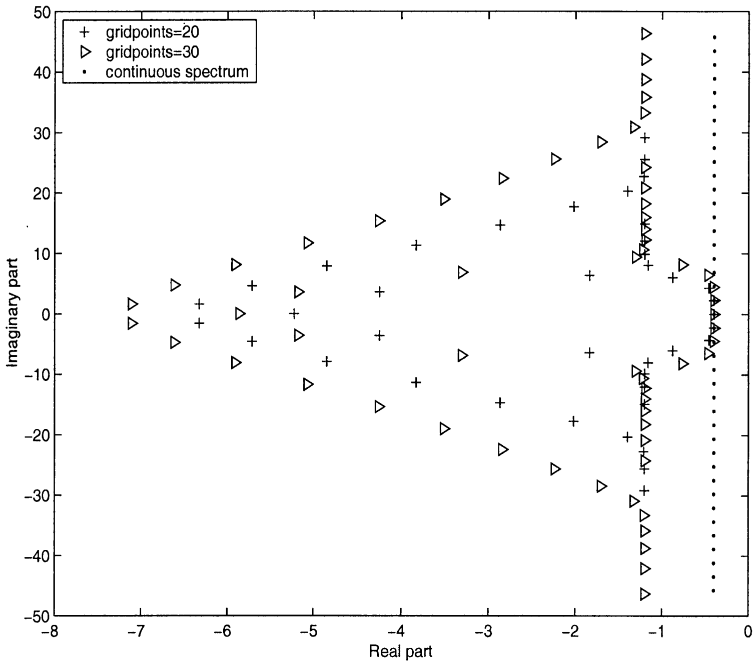


Figure 2. Skew-symmetric spectrum of G, second order case. $a = 1+0.8x, b = -1+0.8x$

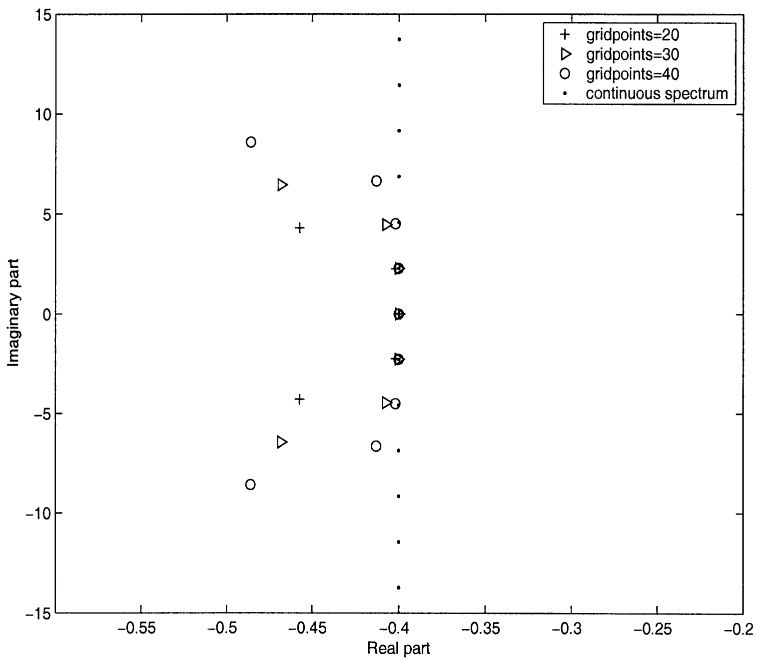


Figure 3. Conservative spectrum of G, second order case. $a = 1 + 0.8x$, $b = -1 + 0.8x$

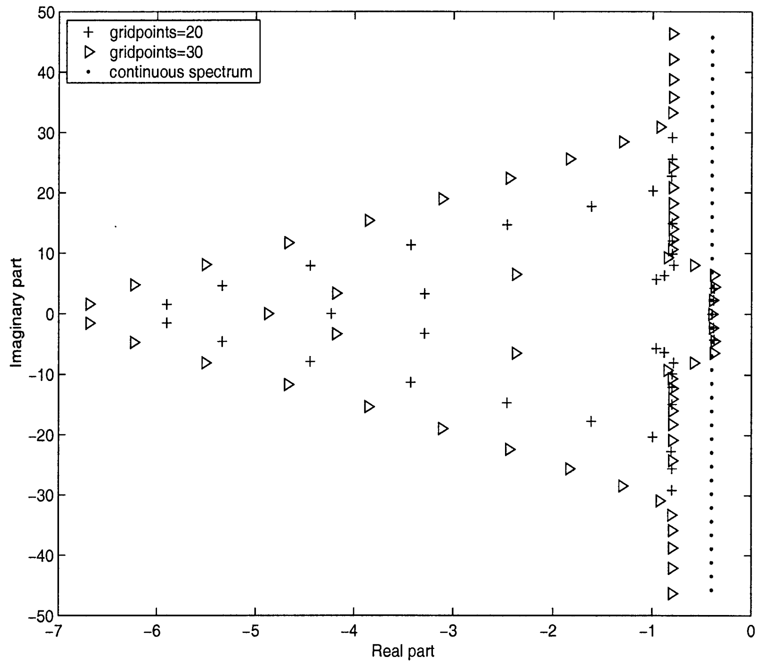
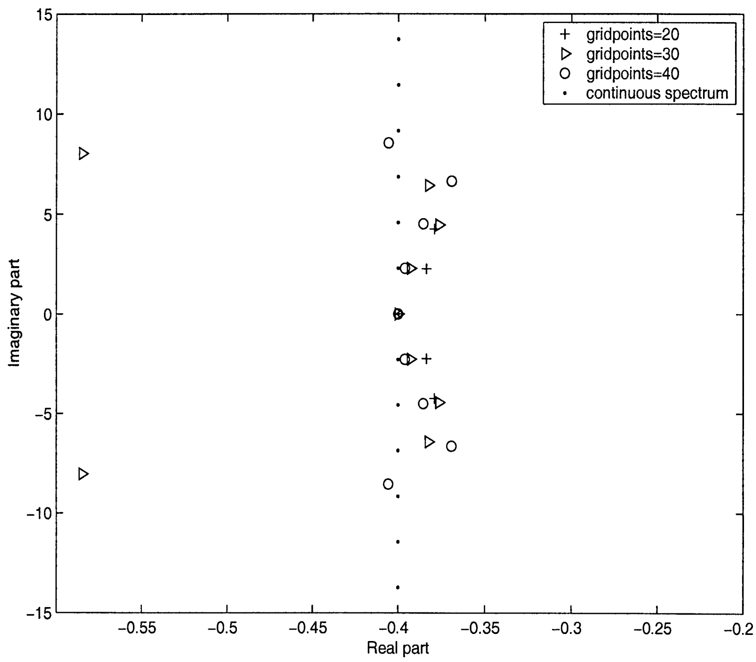


Figure 4. Conservative spectrum of G, second order case. $a = 1 + 0.8x$, $b = -1 + 0.8x$



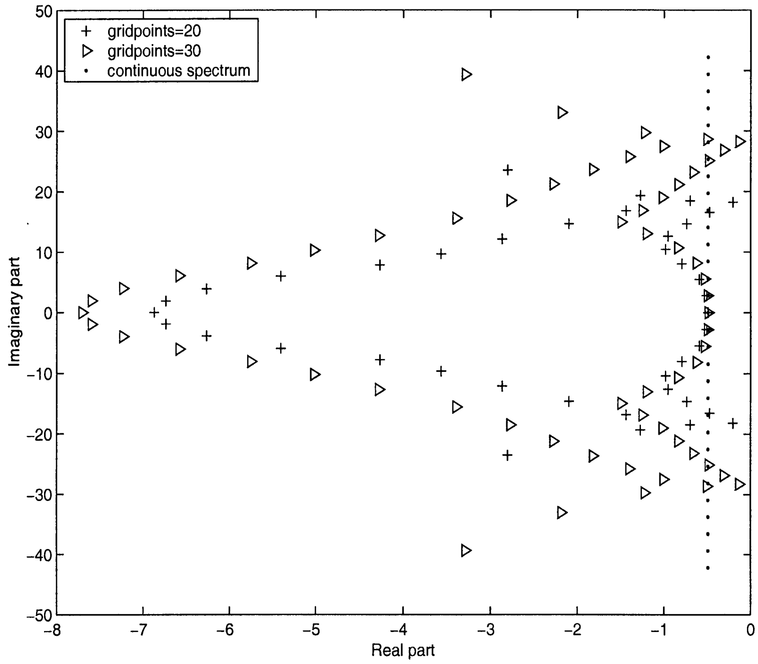


Figure 5. Skew-symmetric spectrum of G, second order case.$a = 1 + 0.8x^4$, $b = -1 + 0.8x^4$

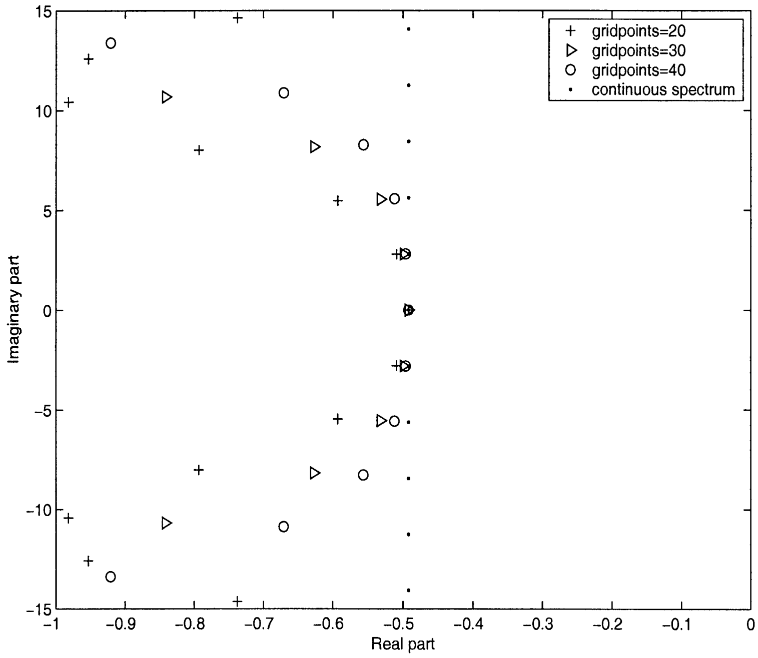


Figure 6. Skew-symmetric spectrum of G, second order case.$a = 1 + 0.8x^4$, $b = -1 + 0.8x^4$

Figure 7. Skew-symmetric spectrum of G, fourth order case. $a = 1 + 0.8sin(7.9x)$, $b = -1 + 0.8sin(7.9x)$.

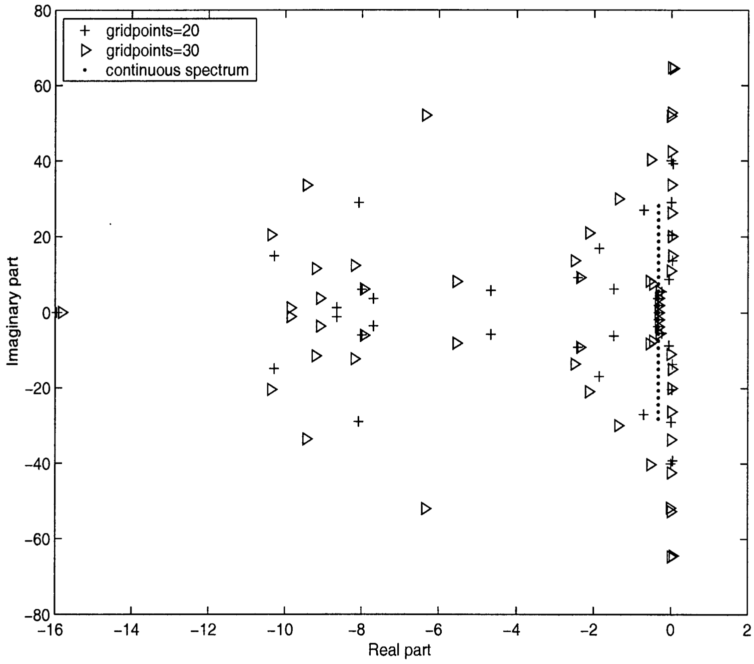
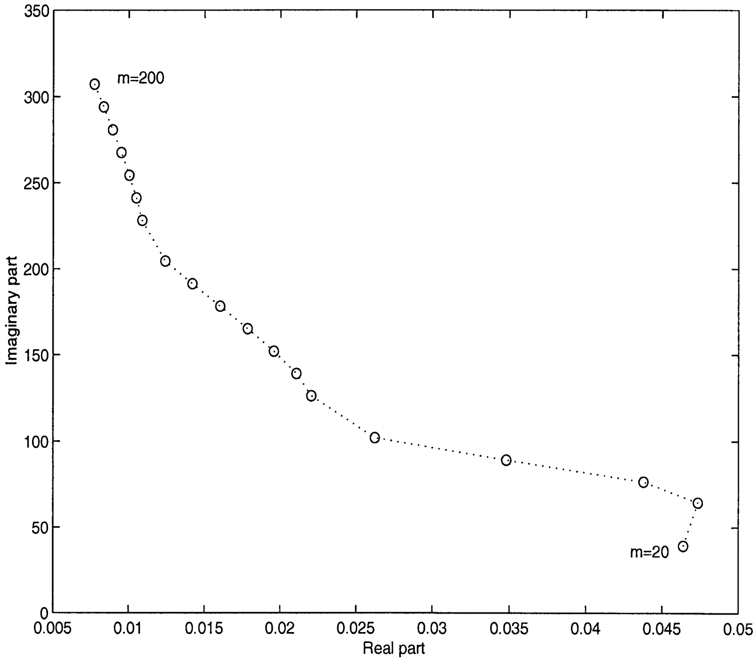


Figure 8. Skew-symmetric spectrum of G, fourth order case. $a = 1 + 0.8sin(7.9x)$, $b = -1 + 0.8sin(7.9x)$. Shows $(real(\lambda))_{max}$ for each grid, m=20:10:200



where

$$G = -\mathcal{DF} + \mathcal{P}^{-1}S,$$

for the conservative formulation and

$$G = -\frac{1}{2}\mathcal{F}_x - \frac{1}{2}(\mathcal{DF} + \mathcal{FD}) + \mathcal{P}^{-1}S,$$

for the skew-symmetric case.

In figures 1 and 2 all eigenvalues converge towards the continuous spectrum from the left.This imply strict stability for the skew-symmetric method in the linear case. The discrete spectrum in figures 3 - 6, for both methods do converge to the continuous spectrum $\lambda_C^{Re} = s$ when refining the grid, but $(\lambda_j^{Re})_{max} \quad j = 1, \dots, N+1$ increases when Δx decreases. This means that neither of these methods can be said to be strictly stable, see definition 3, equation (10). In figure 7 some eigenvalues have real parts greater than zero, which will result in an explosion of the solution as time increases. However, in this case $(\lambda_j^{Re})_{max}$ seems to converge to zero when refining the grid, see figure 8.

We cannot always guarantee strict stability for the skew-symmetric method. However,

$$\frac{d}{dt}||\mathbf{w}||^2_{\mathcal{P}} \quad = \quad -(\mathcal{F}_x\mathbf{w}, \mathbf{w})_{\mathcal{P}} \quad \leq \quad |\mathcal{F}_x|_{max}||\mathbf{w}^2_{\mathcal{P}}||, \tag{25}$$

is always valid, i.e we can obtain an energy estimate. For the conservative scheme (13) we cannot produce an estimate like (25).

2.2.6 Conservative method with artificial dissipation

Although we now have a numerical method with a bounded energy rate, see section 2.2.3, it is preferable to numerically solve the problem (1) using a conservative formulation of the problem, as in (11), especially when the solution is non-smooth and a correct shock speed (in the non linear case) is required [1]. Adding and subtracting $\frac{1}{2}(A_x\mathbf{u} + A\mathcal{D}\mathbf{u})$ and $\frac{1}{2}(B_x\mathbf{v} + B\mathcal{D}\mathbf{v})$ respectively from the conservative formulation (11) we get

$$\begin{aligned} \mathbf{u}_t + DA\mathbf{u} + \frac{1}{2}(A_x\mathbf{u} + AD\mathbf{u}) - \frac{1}{2}(A_x\mathbf{u} + AD\mathbf{u}) &= 0 \\ \mathbf{v}_t + DB\mathbf{v} + \frac{1}{2}(B_x\mathbf{v} + BD\mathbf{v}) - \frac{1}{2}(B_x\mathbf{v} + BD\mathbf{v}) &= 0. \end{aligned} \tag{26}$$

After rearranging the terms in (26) and by using (12) the semi-discrete problem can be written,

$$\mathbf{w}_t + \frac{1}{2}(\mathcal{F}_x\mathbf{w} + \mathcal{DF}\mathbf{w} + \mathcal{FD}\mathbf{w}) = \frac{1}{2}(-\mathcal{DF}\mathbf{w} + \mathcal{F}_x\mathbf{w} + \mathcal{FD}\mathbf{w}), \tag{27}$$

which is exactly the skew-symmetric formulation (17), except for the term on the right hand side in (27) that we denote by R. If R is a non dissipative term we need to dominate it by suitable artificial dissipation terms, preferably without affecting the order of accuracy for the numerical problem. If we accomplish that, we have a conservative formulation of the problem, that unlike (11) leads to an energy estimate.

For the first equation in (26), R has the form

$$R_i = \frac{1}{2}(-DA\mathbf{u} + A_x\mathbf{u} + AD\mathbf{u})_i, \quad i = \text{the i:th node point.} \tag{28}$$

The size and form of the artificial dissipation term depends on the spatial difference operator, D. In this paper we use second-, fourth- and sixth order accurate central difference operators (see Appendix A) for a description of the first derivative operators. Let us start by considering the second order case, we get,

$$DF \approx F_x + \frac{(\Delta x)^2}{6}F_{xxx} + O\left((\Delta x)^4\right).$$

This implies that

$$R_i = \frac{1}{2}\left[-D(au) + aDu + (Da)u\right]_i \approx -\frac{(\Delta x)^2}{4}\left[(a_x u_x)_x\right]_i. \tag{29}$$

The contribution to the energy estimate of R_i can be estimated by interpreting the result in continuous frame and by multiplying (29) by u and integrate in space. We get

$$\int_0^1 uRdx = BT + \frac{(\Delta x)^2}{4}\int_0^1 a_x u_x^2 dx, \tag{30}$$

where BT stands for the (neglected) boundary terms. According to (30) R_i is of a dissipative nature if $a_x < 0$. To make sure that our artificial dissipation (added to the right hand side of equation (13)) is dissipative and large enough to balance R, we use the dissipation operators developed in [6] and write this term as

$$DI = -\frac{(\Delta x)^2}{4}\widetilde{P}^{-1}D_1^T(|A_x|_{max}C)D_1\mathbf{u}. \tag{31}$$

In (31) D_1 approximates a first derivative, C is a diagonal matrix that reduces the values of $|A_x|_{max}$ at the boundaries and $\widetilde{P}^{-1} = \Delta x P^{-1}$ is included in order to obtain the correct discrete energy estimate, see [6].

The next case to consider is the fourth order operator. Taylor expansion yields,

$$DF \approx F_x + \frac{(\Delta x)^4}{30}F_{xxxxx} + O\left((\Delta x)^6\right).$$

This leads to, see (28)

$$R_i \approx \frac{(\Delta x)^4}{12}\left[(a_{xxx}u_x)_x + (a_x u_{xx})_{xx}\right]_i.$$

The contribution to the energy rate of R_i becomes,

$$\int_0^1 uRdx = BT - \frac{(\Delta x)^4}{12}\int_0^1 a_{xxx}u_x^2dx + \frac{(\Delta x)^4}{12}\int_0^1 a_x u_{xx}^2 dx. \quad (32)$$

The first integral in (32) is negative for $a_{xxx} > 0$ and the second one for $a_x < 0$. The dissipation term we will use becomes

$$DI = -\frac{(\Delta x)^4}{12}\widetilde{P}^{-1}\left(D_1^T(|A_{xxx}|_{max}C)D_1\mathbf{u} + D_2^T(|A_x|_{max}C)D_2\mathbf{u}\right). \quad (33)$$

D_2 in (33) approximates a second derivative.

Finally we consider the sixth order case. Taylor expansion gives,

$$DF \approx F_x + \frac{(\Delta x)^6}{140}F_{xxxxxxx} + O\left((\Delta x)^8\right). \quad (34)$$

The relation (34) and (28) implies,

$$R_i \approx -\frac{(\Delta x)^6}{40}\left[(a_{xxxxx}u_x)_x + 2\left(a_{xxx}u_{xx}\right)_{xx} + \left(a_x u_{xxx}\right)_{xxx}\right]_i$$

The contribution to the energy estimate related to R is

$$\begin{aligned}\int_0^1 uRdx &= BT + \frac{(\Delta x)^6}{40}\int_0^1 a_{xxxxx}u_x^2dx - \\ &\frac{(\Delta x)^6}{20}\int_0^1 a_{xxx}u_{xx}^2dx + \frac{(\Delta x)^6}{40}\int_0^1 a_x u_{xxx}^2dx.\end{aligned}$$

The dissipation term is now determined to be

$$\begin{aligned}DI &= -\frac{(\Delta x)^6}{40}\widetilde{P}^{-1}\big(D_1^T(|A_{xxxxx}|_{max}C)D_1\mathbf{u} + \\ &2D_2^T(|A_{xxx}|_{max}C)D_2\mathbf{u} + D_3^T(|A_x|_{max}C)D_3\mathbf{u}\big)\end{aligned}$$

where D_3 approximates a third derivative.

In figures 9 and 10, the dissipative term has obviously balanced R enough, see (28) and compare with figures 5 and 6. Not only do the discrete eigenvalues converge to the continuous ones, but also $(\lambda_j^{Re})_{max}$ $j = 1, \ldots, N+1$, converges as Δx goes to zero. All eigenvalues in figure 9 converge from the left, except for the one with imaginary part equal to zero. Artificial dissipation decreases the energy rate, but do not guarantee that λ_j^{Re} of G in (24) decreases, except for the ones belonging to the symmetric part of G.

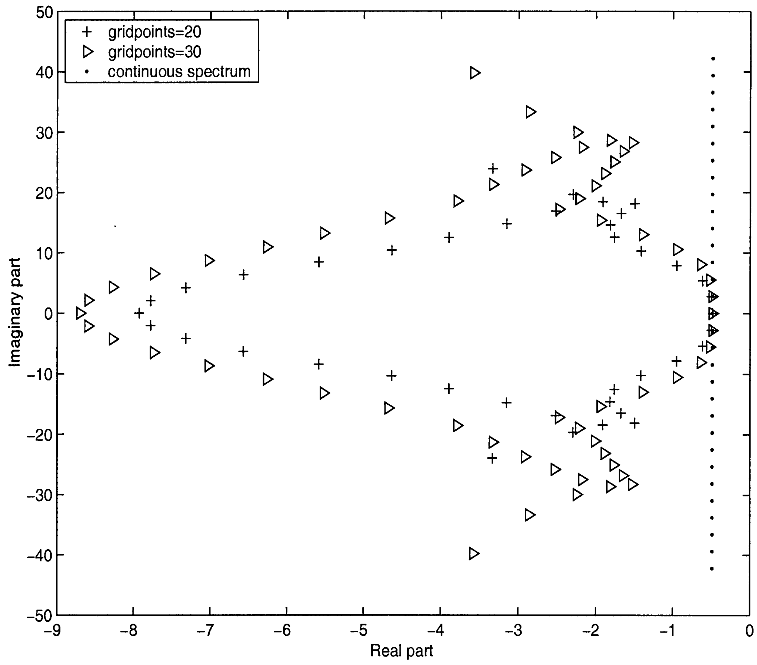


Figure 9. Spectrum of G, conservative method with dissipation term. Second order case. $a = 1 + 0.8x^4$, $b = -1 + 0.8x^4$

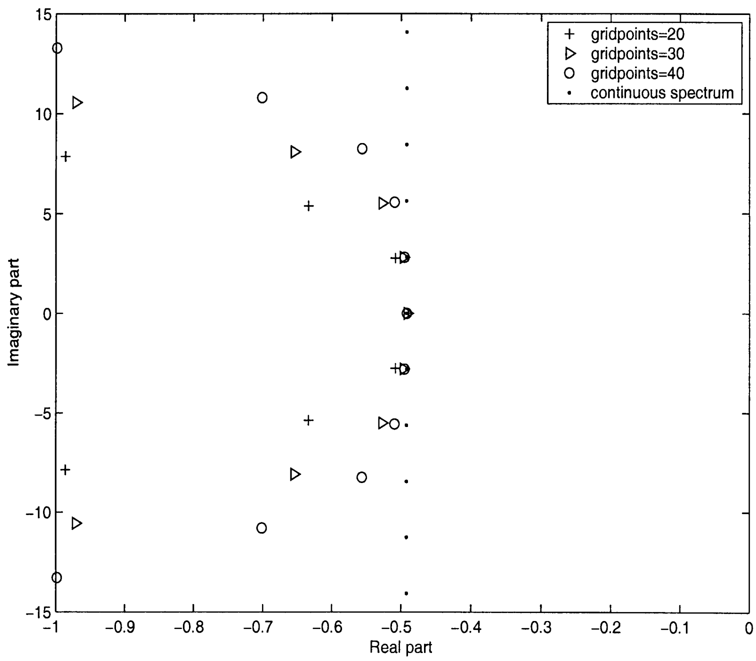


Figure 10. Spectrum of G, conservative method with dissipation term. Second order case. $a = 1 + 0.8x^4$, $b = -1 + 0.8x^4$

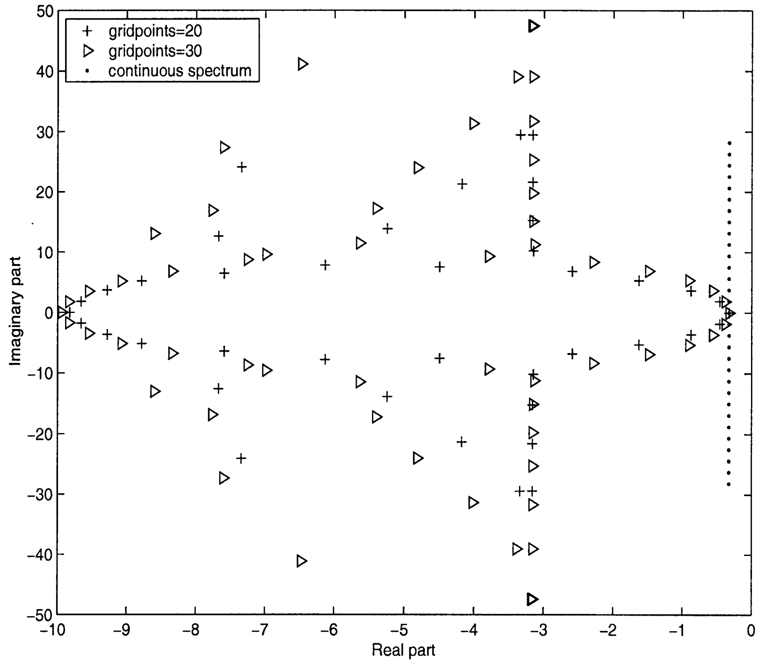


Figure 11. Spectrum of G, conservative method with dissipation term. Second order case. $a = 1 + 0.8sin(7.9x)$, $b = -1 + 0.8sin(7.9x)$

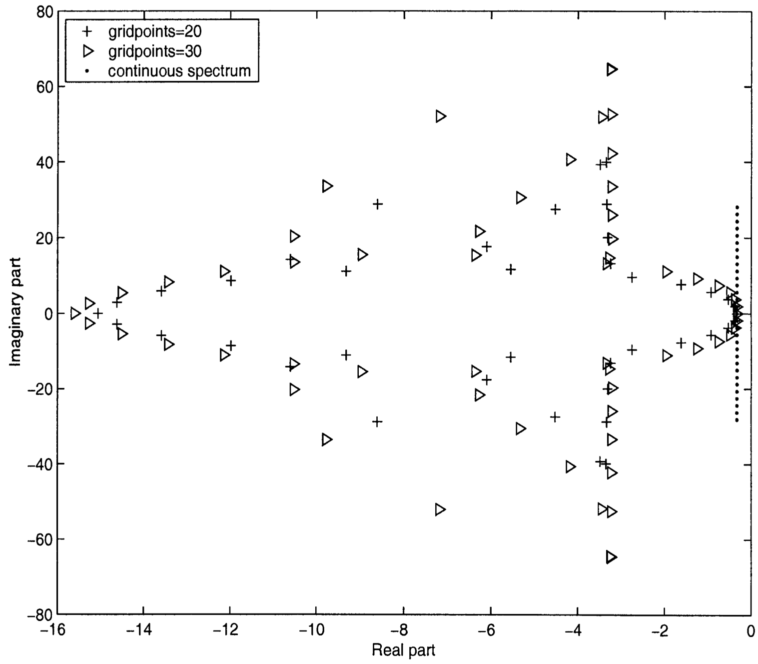


Figure 12. Spectrum of G, conservative method with dissipation term. Fourth order case. $a = 1 + 0.8sin(7.9x)$, $b = -1 + 0.8sin(7.9x)$

Figure 13. Spectrum of G, conservative method with dissipation term. Second order case. $a = 1 + 0.8sin(7.9x)$, $b = -1 + 0.8sin(7.9x)$

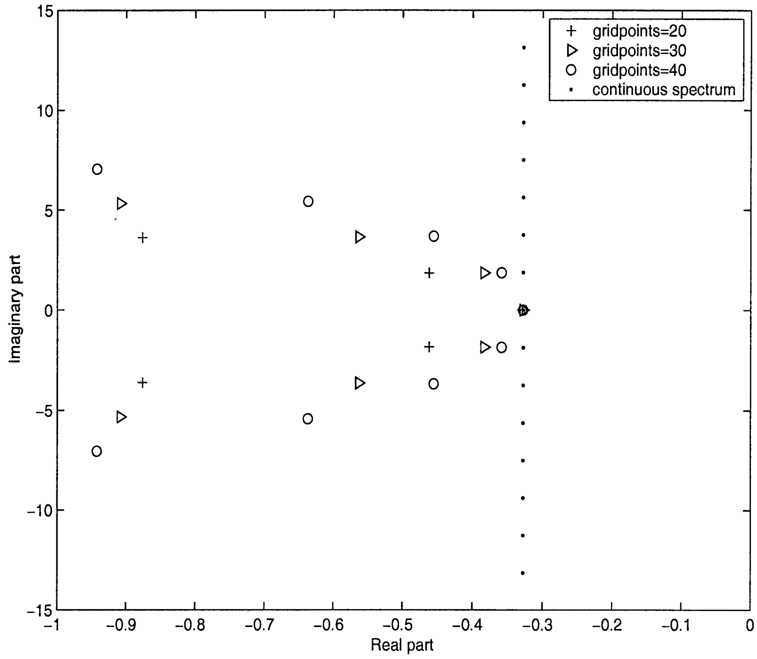


Figure 14. Spectrum of G, conservative method with dissipation term. Fourth order case. $a = 1 + 0.8sin(7.9x)$, $b = -1 + 0.8sin(7.9x)$

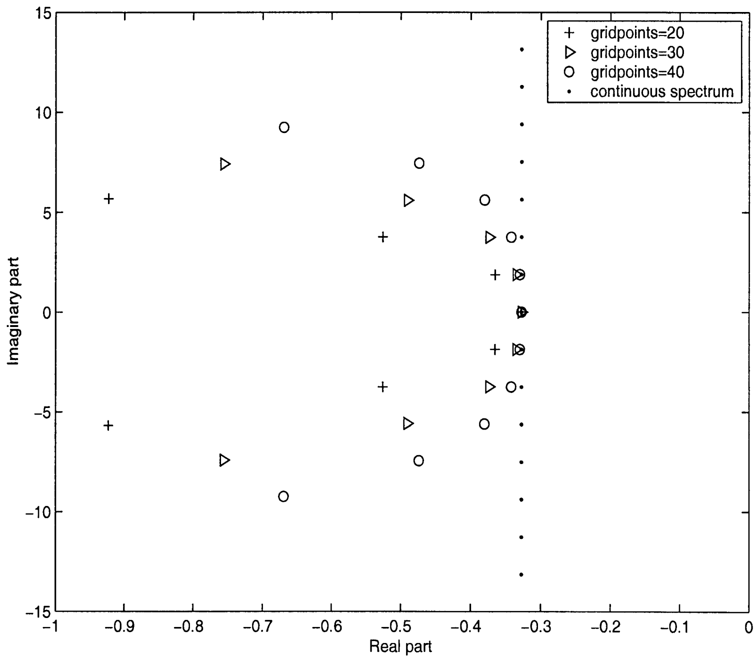
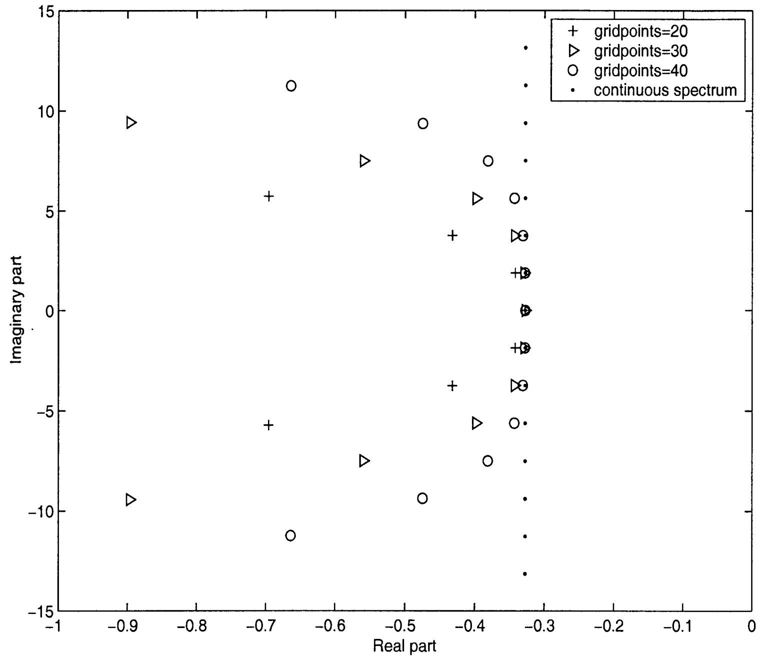


Figure 15. Spectrum of G, conservative method with dissipation term. Sixth order case. $a = 1 + 0.8sin(7.9x)$, $b = -1 + 0.8sin(7.9x)$



As can be seen in figures 11 and 12 a higher order of accuracy does not mean that all eigenvalues are closer to the continuous ones. But they do converge faster, which is shown in figures 13, 14 and 15 . Eigenvalues for $a(x)$ and $b(x)$ beeing polynomials up to fifth degree as well as trigonometric functions have been computed. In all these cases $(\lambda_j^{Re})_{max}$ $j = 1, \ldots, N+1$, converges as Δx goes to zero for the conservative approximation with the new artificial dissipation, although they sometimes converge from the right.

3 The nonlinear problem

3.1 The continuous case

In this section we consider the nonlinear Burger's equation.

$$u_t + \frac{1}{2}(u^2)_x = \epsilon u_{xx}. \qquad (35)$$

In (35), ϵ is a small positive number. Writing (35) on a skew-symmetric formulation, we get

$$u_t + (1-\gamma)\frac{1}{2}(u^2)_x + \gamma u u_x = \epsilon u_{xx} \qquad (36)$$

To obtain an energy estimate for the discrete problem we must use $\gamma = \frac{1}{3}$. The energy estimate for (36) becomes

$$\frac{d}{dt}||u||^2_{\ell_2} = 2(\epsilon u u_x - \frac{1}{3}u^3)\Big|_0^1 - 2\epsilon\int_0^1 u_x^2 dx$$

3.2 The discrete case

The semi-discrete skew-symmetric problem

$$\mathbf{u}_t + \frac{1}{3}UP^{-1}Q\mathbf{u} + \frac{1}{3}P^{-1}QU\mathbf{u} = \epsilon P^{-1}QP^{-1}Q\mathbf{u}, \qquad (37)$$

where $U = diag(\mathbf{u})$, yields the following energy rate

$$\frac{d}{dt}||\mathbf{u}||^2_{\mathcal{P}} \quad = \quad -\frac{2}{3}\mathbf{u}^T BU\mathbf{u} + 2\epsilon\mathbf{u}^T BD\mathbf{u} - 2\epsilon\,(D\mathbf{u}, D\mathbf{u})_P\,. \qquad (38)$$

Equation (38) correspond exactly to the continuous energy rate. Let $a = \frac{u}{2}$ in (35). We then determine the dissipation terms for the different SBP-operators, see appendix A, in the same way as in section 2.2.6, omitting the term on the right in (37) since it is already dissipative. The dissipation terms turn out be $\frac{2}{3}$ times the ones calculated in section 2.2.6. The difference is due to the fact that we use $\gamma = \frac{1}{3}$ here and $\gamma = \frac{1}{2}$ in section 2.2.6.

4 Numerical experiments

We have constructed a strictly stable method using a conservative formulation augmented with suitable artificial dissipation. In this section we test our scheme and see how it performs on various problems. In addition, we want to verify that no loss of accuracy occurs.

4.1 The linear case

Consider the problem (11). To integrate in time we use a fourth order accurate Runge-Kutta method. In this paper we have choosen to investigate a few cases where we are able to determine the solution analytically, see appendix B. Since a_x and b_x, used in figures 16, 17 and 18 alternate be-

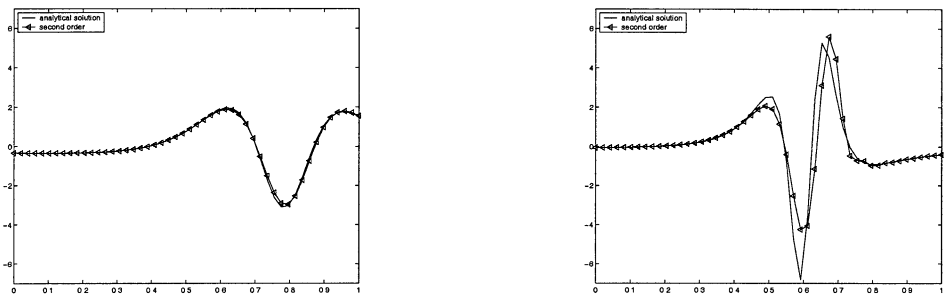


Figure 16. v at $t = .5$ and $t = 1.1$. Second order case. $a = 1 + .8sin(\pi x)$, $b = -1 + .8sin(\pi x)$,$f = sin(2\pi x)$, $g = -f$, $N = 50$ and $k = .001$

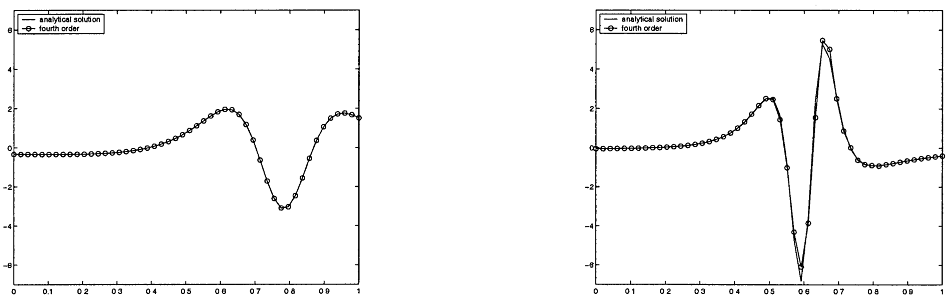


Figure 17. v at $t = .5$ and $t = 1.1$. Fourth order case.$a = 1 + .8sin(\pi x)$, $b = -1 + .8sin(\pi x)$,$f = sin(2\pi x)$,$g = -f$, $N = 50$ and $k = .001$

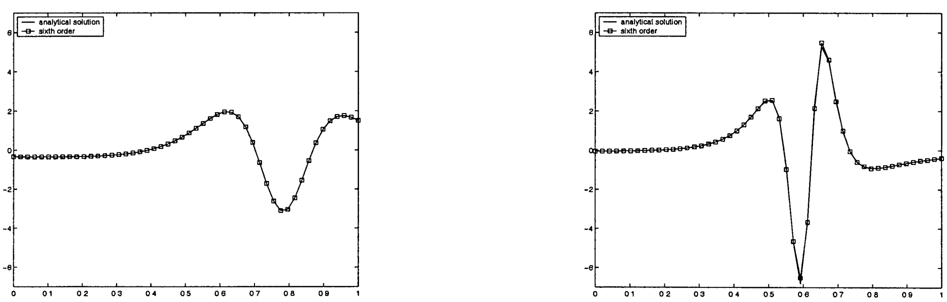
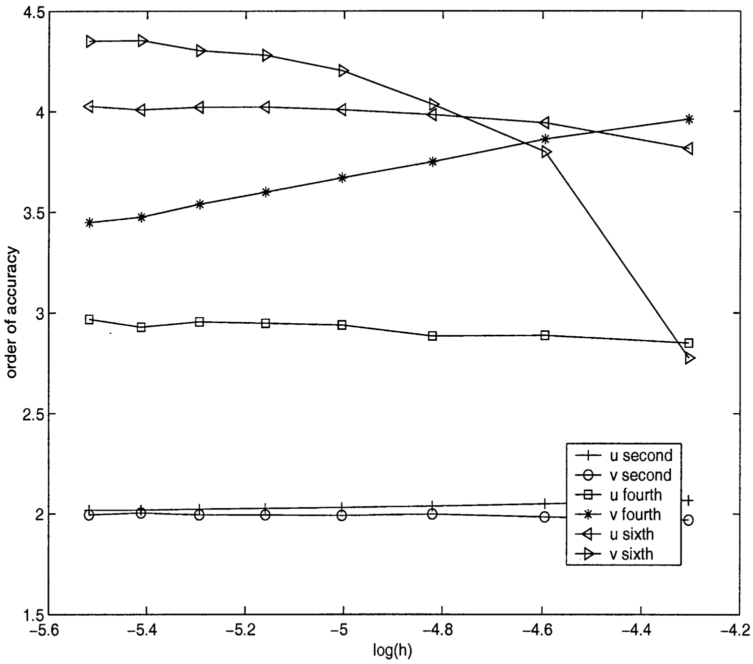


Figure 18. v at $t = .5$ and $t = 1.1$. Sixth order case.$a = 1 + .8sin(\pi x)$, $b = -1 + .8sin(\pi x)$,$f = sin(2\pi x)$,$g = -f$, $N = 50$ and $k = .001$

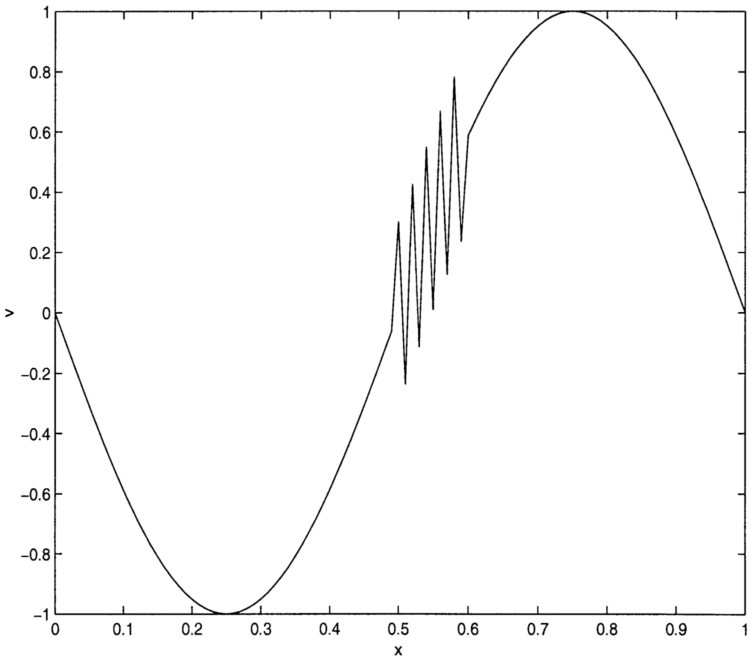
tween positive and negative numbers, the amplitude of the solution might grow in time, see equation (5).

Figure 19. Error at t=.2 sek. $a = 1 + .8sin(\pi x)$,$b = -1 + .8sin(\pi x)$,$f = sin(2\pi x)$,$g = -f$ and k=.0001



Since SBP-operators with diagonal norms are used, the order of accuracy at the boundary is half the one used in the interiour of the domain, see [7]. This implies that the total order of accuray in space becomes 2, 3 and 4 for the second, fourth and sixth order schemes respectively. The order of accuary is not altered by the artificial dissipation, which can be seen in figure 19.

Figure 20. v at $t = 0$. Sixth order case. $a = 1 + .8sin(\pi x)$, $b = -1 + .8sin(\pi x)$, $f = sin(2\pi x)$,$g = -f +$ some perturbation, $N = 101$ and $k = .001$



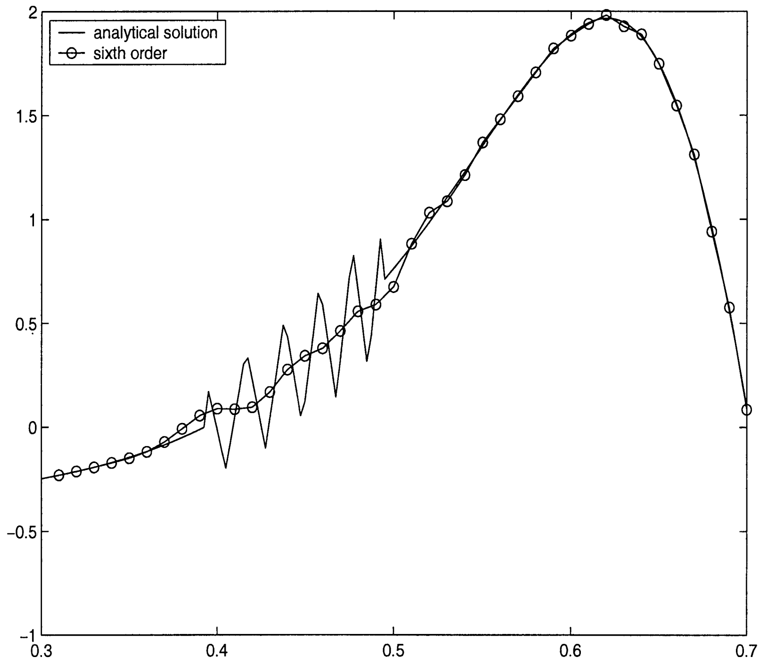


Figure 21. v at $t = 0.5$. Sixth order case.$a = 1 + .8sin(\pi x)$, $b = -1 + .8sin(\pi x)$,$f = sin(2\pi x)$,$g = -f +$ some perturbation, $N = 101$ and $k = .001$

In figures 20 and 21, an initial perturbation in v decreases and finally vanishes as t increases. The disturbance is removed by the dissipation terms. In the continuous case, the perturbation propagates along with the rest of the solution.

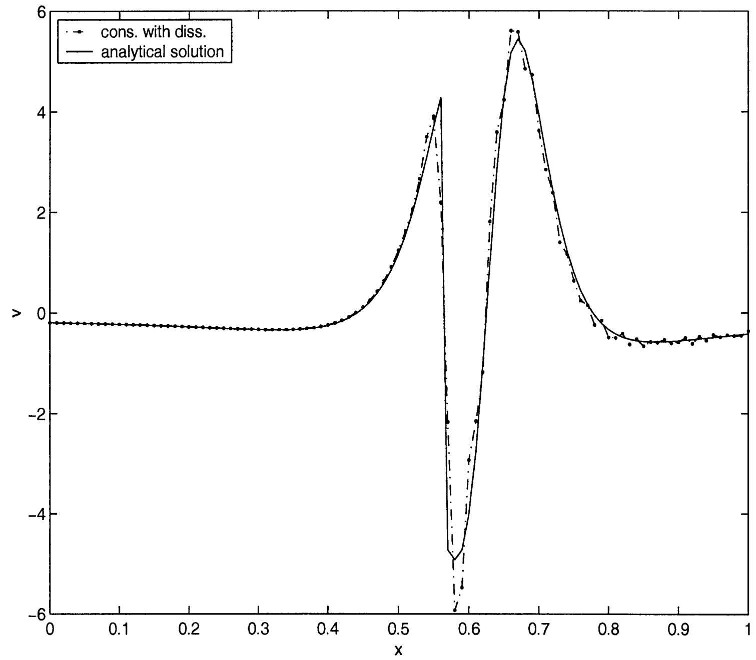
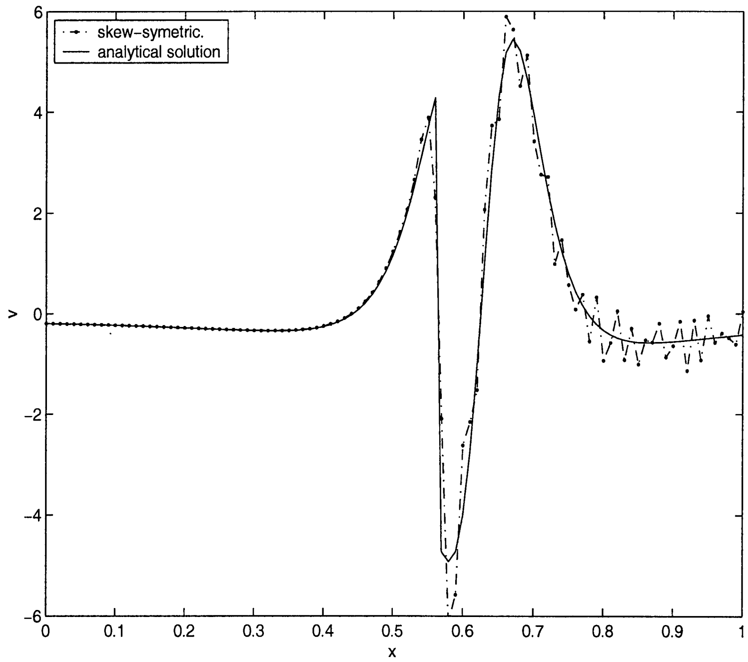


Figure 22. v at $t = 1.0$. Sixth order case.$a = 1 + .8sin(\pi x)$, $b = -1 + .8sin(\pi x)$, $f = sin(1.5\pi x)$,$g = -f$, $N = 100$ and $k = .001$

By choosing $f = sin(1.5\pi x)$ and $g = -f$ we get a discontinuity that travels through the right boundary into the solution of v. In figures 22 and 23 we see that the conservative method with artificial dissipation reduces the perturbations caused by the discontinuity more than the skew-symetric method.

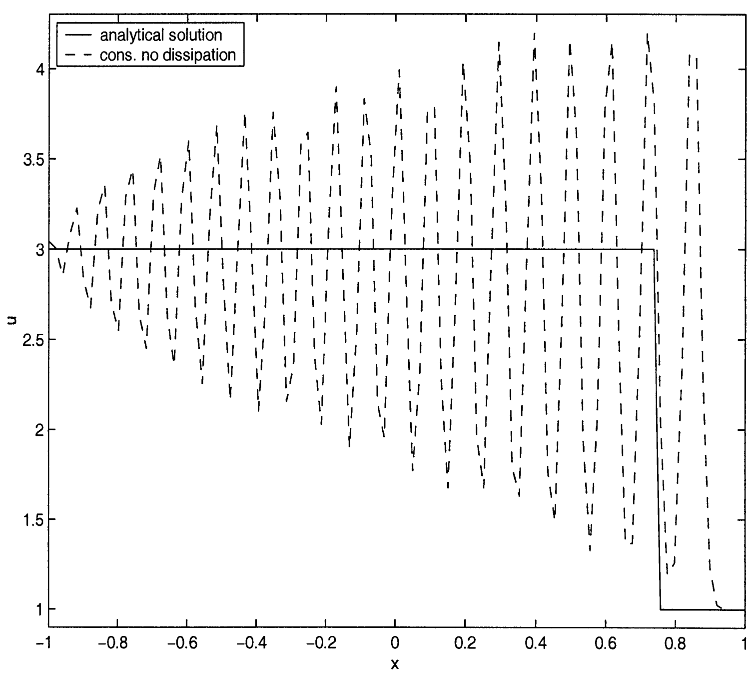
Figure 23. v at $t = 1.0$. Sixth order case. $a = 1 + .8sin(\pi x)$, $b = -1 + .8sin(\pi x)$, $f = sin(1.5\pi x)$, $g = -f$, $N = 100$ and $k = .001$



4.2 The nonlinear case

We consider Burger's equation (35). The initial condition is such that the analytical solution is a shock wave (or sharp gradient) propagating to the right. The solution is constant on both sides of the shock wave, but with different values. The steepness of the shock is depending on the value of ϵ.

Figure 24. u at $t = 0.9$. Second order case without dissipation. $\epsilon = 10^{-8}$, $N = 100$ and $k = .0002$



Note that figures 24 and 25 show that the artificial dissipation improves the numerical solution by reducing the oscillations behind the shock. In figures 25 and 26 we see that the elimination of the oscillations is further improved

Figure 25. u at $t = 0.9$. Second order case. $\epsilon = 10^{-8}$, $N = 100$ and $k = .0002$

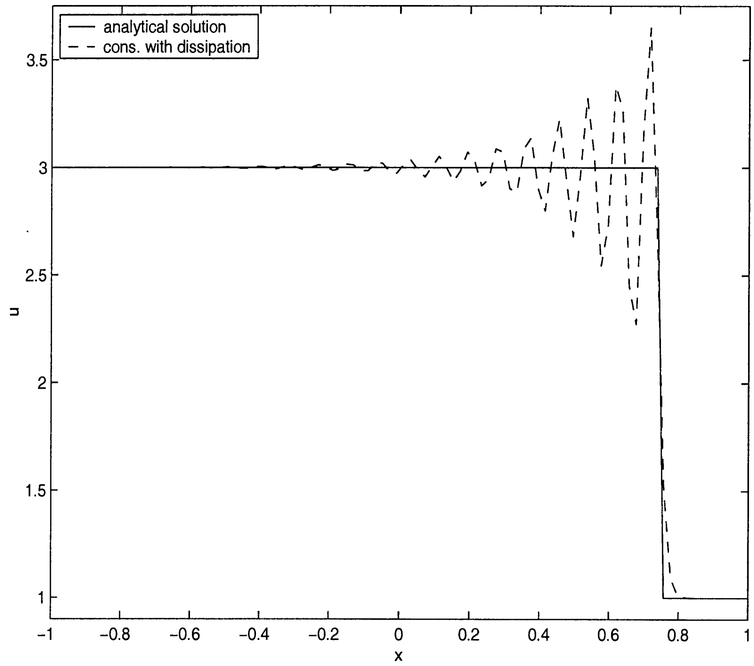
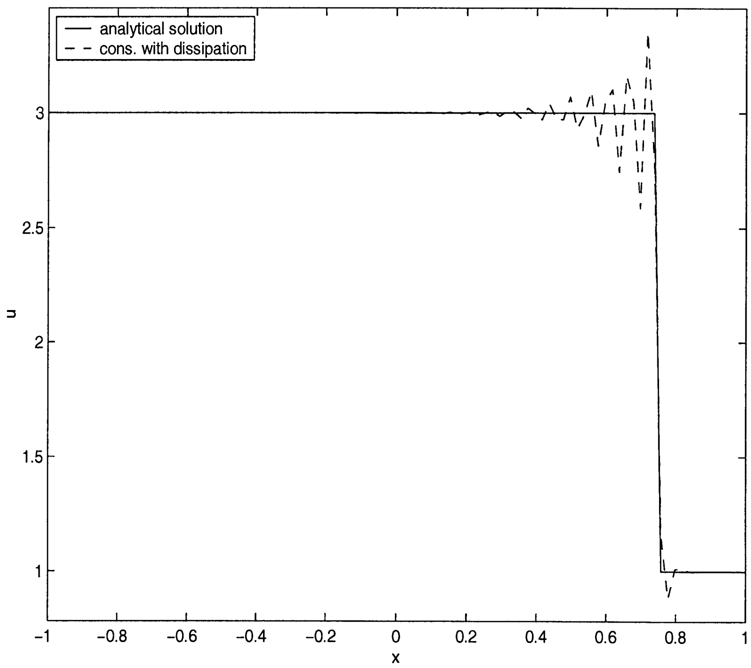


Figure 26. u at $t = 0.9$. Sixth order case. $\epsilon = 10^{-8}$, $N = 100$ and $k = .0002$



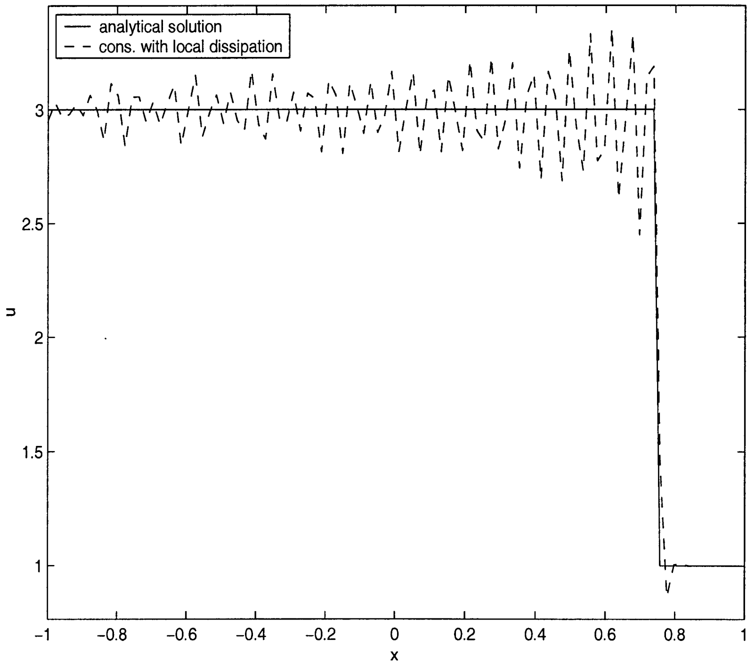


Figure 27. u at $t = 0.9$. Sixth order case with local artificial dissipation. $\epsilon = 10^{-8}$, $N = 100$ and $k = .0002$

by using a high order method. Using local artificial dissipation, see figure 27, results in a solution where the oscillations are more scattered. With local artificial dissipation, we mean that the amount of dissipation applied in each node depends on derivatives in that specific node, i.e. $|A_x|_{max}$ in (31) is replaced by $|A_x|$. Since SBP-operators with diagonal norms are

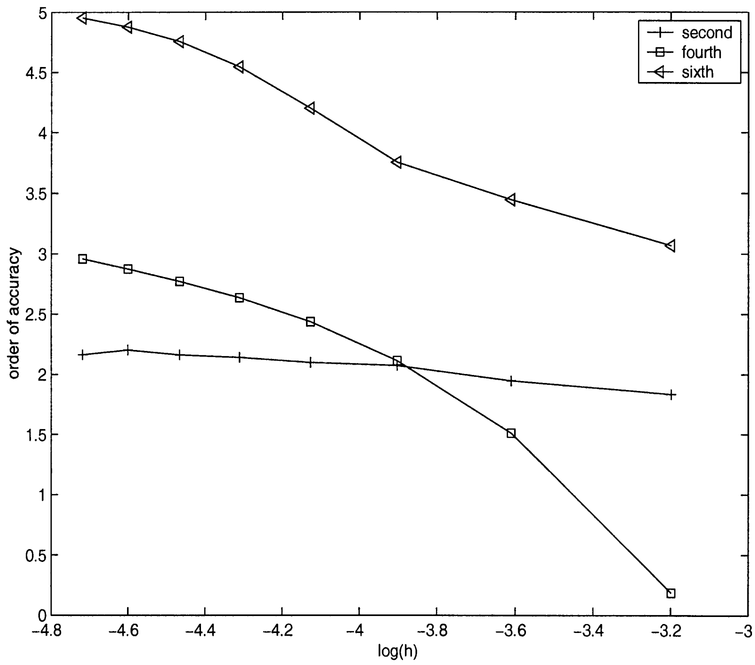


Figure 28. Error at $t = 0.002$. Nonzero derivatives at the left boundary. $\epsilon = 0.05$, $k = .00001$ and $x \in [-1\ 1]$

used, the order of accuracy in space decreases to 2, 3 and 4 respectively (however, for some reason the order of accuracy becomes 5 for the sixth order case in figure 28).

5 Conclusions

We have determined a new type of artificial dissipation that depends on the variable coefficients (or the solution) and it's derivatives and the size of the grid. We have shown that it is possible to make the conservative method strictly stable by adding an artificial dissipation term without destroying the accuracy. Strict stability cannot be obtained using the skew-symmetric method.

Tests have shown that the conservative method combined with our artificial dissipation terms are capable of eliminating perturbations, due to for instance shock waves. Presumably it works even better for nonlinear problems with nonzero gradients. However, the amount of artificial dissipation is probably not optimal.

In this paper we have used global dissipation, which means that the same amount of dissipation is added in each node. A possible improvement of this procedure would be to allow amount of dissipation to vary locally. This is important since the amount of artificial dissipation needed often varies over the domain.

Appendix A

SBP-operators

A SBP-operator with diagonal norm [7] is on the form

$$D = \frac{1}{\Delta x}\begin{pmatrix} A & & & & \\ & d & & & \\ & & \ddots & & \\ & & & d & \\ & & & & B \end{pmatrix}$$

$A = A(n,m)$ is a matrix who takes care of derivatives close to the left boundary. $B = -rot(A,180)$ is of the same size as A but deals with derivatives on the right boundary. The value of n and m depends on the order of accuracy of the SBP-operator. The derivatives in the inner is taken care of by the row vector d. d:s wideness depends on what SBP-operator we use.

A SBP-operator can be written as $D = P^{-1}Q$, where P in our case is a diagonal matrix.

Second order accurate difference operator:

$$A = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

$$d = \frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$$

$$P = \Delta x \ \text{diag}\begin{pmatrix} \frac{1}{2} & 1 & \dots & 1 & \frac{1}{2} \end{pmatrix}$$

Fourth order accurate difference operator:

$$A = \begin{pmatrix} -\frac{24}{17} & \frac{59}{34} & -\frac{4}{17} & -\frac{3}{34} & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{4}{43} & -\frac{59}{86} & 0 & \frac{59}{86} & -\frac{4}{43} & 0 \\ \frac{3}{98} & 0 & -\frac{59}{98} & 0 & \frac{32}{49} & -\frac{4}{49} \end{pmatrix}$$

$$d = \frac{1}{12}\begin{pmatrix} 1 & -8 & 0 & 8 & -1 \end{pmatrix}$$

$$P = \Delta x \ \text{diag}\begin{pmatrix} \frac{17}{48} & \frac{59}{48} & 1 & \dots & 1 & \frac{43}{48} & \frac{49}{48} \end{pmatrix}$$

Sixth order accurate difference operator:

$$A_{1:6,1:3} = \begin{pmatrix} -\frac{21600}{13649} & \frac{43200}{13649}c - \frac{7624}{40947} & -\frac{172800}{13649}c + \frac{715489}{81894} \\ -\frac{8640}{12013}c + \frac{7624}{180195} & 0 & \frac{86400}{12013}c - \frac{57139}{12013} \\ \frac{17280}{2711}c - \frac{715489}{162660} & -\frac{43200}{2711}c + \frac{57139}{5422} & 0 \\ -\frac{25920}{5359}c + \frac{187917}{53590} & \frac{86400}{5359}c - \frac{745733}{64308} & -\frac{86400}{5359}c + \frac{176839}{16077} \\ \frac{34560}{7877}c - \frac{147127}{47262} & -\frac{129600}{7877}c + \frac{91715}{7877} & \frac{172800}{7877}c - \frac{242111}{15754} \\ -\frac{43200}{43801}c + \frac{89387}{131403} & \frac{172800}{43801}c - \frac{240569}{87602} & -\frac{259200}{43801}c + \frac{182261}{43801} \end{pmatrix}$$

$$A_{1:6,4:6} = \begin{pmatrix} \frac{259200}{13649}c - \frac{187917}{13649} & -\frac{172800}{13649}c + \frac{735635}{81894} & \frac{43200}{13649}c - \frac{89387}{40947} \\ -\frac{172800}{12013}c + \frac{745733}{72078} & \frac{129600}{12013}c - \frac{91715}{12013} & -\frac{34560}{12013}c + \frac{240569}{120130} \\ \frac{86400}{2711}c - \frac{176839}{8133} & -\frac{86400}{2711}c + \frac{242111}{10844} & \frac{25920}{2711}c - \frac{182261}{27110} \\ 0 & \frac{43200}{5359}c - \frac{165041}{32154} & -\frac{17280}{5359}c + \frac{710473}{321540} \\ -\frac{86400}{7877}c + \frac{165041}{23631} & 0 & \frac{8640}{7877}c \\ \frac{172800}{43801}c - \frac{710473}{262806} & -\frac{43200}{43801}c & 0 \end{pmatrix}$$

$$A_{1:6,7:9} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{72}{5359} & 0 & 0 \\ -\frac{1296}{7877} & \frac{144}{7877} & 0 \\ \frac{32400}{43801} & -\frac{6480}{43801} & \frac{720}{43801} \end{pmatrix}$$

where $c = \frac{342523}{518400}$.

$$d = \frac{1}{60}\begin{pmatrix} -1 & 9 & -45 & 0 & 45 & -9 & 1 \end{pmatrix}$$

$$P = \Delta x \;\; \mathrm{diag}\Big(\frac{13649}{43200} \;\; \frac{12013}{8640} \;\; \frac{2711}{4320} \;\; \frac{5359}{4320} \;\; \frac{7877}{8640} \;\; \frac{43801}{43200} \;\; 1 \;\; \cdots \;\; 1 \;\; \frac{43801}{43200} \;\; \frac{7877}{8640} \;\; \frac{5359}{4320} \;\; \frac{2711}{4320} \;\; \frac{12013}{8640} \;\; \frac{13649}{43200} \Big)$$

Appendix B

Analytical solution

Consider following initial value problem

$$u_t + (au)_x = 0, \quad u(x,0) = f(x) \tag{39}$$

where $a = a(x)$. We can solve (39) by transforming this partial differential equation to three ordinary differential equations. This is done by adding two new variables, y and s, where we define y as $\frac{du}{dy} = -a_x u$ and s as $x = s$ when $t = 0$. The three new equations are

$$\begin{aligned} \frac{dt}{dy} &= 1 \\ \frac{dx}{dy} &= a \\ \frac{du}{dy} &= -a_x u \end{aligned} \tag{40}$$

This gives the solution

$$u = f(s)e^{\int_{y_0}^{y} -a_x dy}$$

where $a = a(x)$, $x = x(y,s)$. s can be determined from

$$\int_0^x \frac{1}{a} dx = y + \int_0^s \frac{1}{a} dx \tag{41}$$

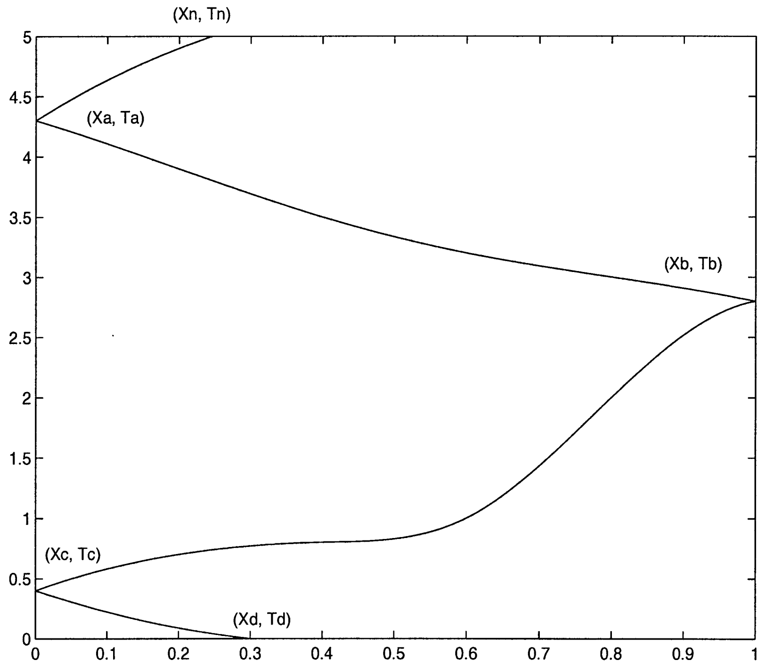
Using the relation $y = t$ from the first equation in (40) the final expression becomes

$$u = f(s)e^{\int_0^t -a_x dy}$$

s can be seen as a characteristic line to a point in the analytical solution. When we express s explicity from (41) we do not always get $s = x$ when $t = 0$, for instance for some periodic $a(x)$. For that reason this solution only holds for some $a(x)$.

In this paper we wish to solve (1), (2), (3). Since $a > 0$ and $b < 0$, points in solution of u move towards the right boundary. And consequently points in the solution of v move in opposite direction. So when we wish to find out the value of u in a certain point, after a certain time, we determine where the point were located at $t = 0$. This is done by following that specific characteristic line back in time to the boundary, where we note the time. Since we now know x and t at the boundary we can track the point through the solution of v to the other boundary. This procedure is repeated until $t = 0$ The value of u in this specific point, for the case in figure 29 is

Figure 29. Characteristic lines for a point $p = (x_n, t_n)$ in u



then

$$\begin{aligned} u(x_n, t_n) &= \alpha v(0, t_a) e^{\int_{t_a}^{t_n} -a_x dy} \\ v(0, t_a) &= \beta u(1, t_b) e^{\int_{t_b}^{t_a} -a_x dy} \\ u(1, t_b) &= \alpha v(0, t_c) e^{\int_{t_c}^{t_b} -a_x dy} \\ v(0, t_c) &= g(s_v(0, t_c)) e^{\int_0^{t_c} -a_x dy} \end{aligned}$$

The final expression is

$$u(x_n, t_n) = \alpha^2 \beta g(s_v(0, t_c)) e^{\int_0^{t_n} -a_x dy}$$

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Abstract An artificial dissipation term for linear and nonlinear hyperbolic Cauchy problems is determined such that we obtain an energy estimate despite a conservative formulation of the problems. The differential equations are solved using second, fourth and sixth order accurate difference operators, which all satisfy summation-by-parts properties. The dissipation terms are computed such that there is no loss of accuracy.		
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Konservativa Differensformuleringar, Energiuppskattningar och Artificiell Dissipation

Sammanfattning

En artificiell dissipation för linjära och icke-linjära hyperboliska Cauchy-problem bestäms så att en energiuppskattning erhålls trots en konservativ formulering av problemen. Differentialekvationerna löses med andra-, fjärde- och sjätte ordningens differensoperatorer, som alla satisfierar "summation-by-parts" egenskaper. Dissipationstermerna bestäms så att noggrannhetsordningen inte påverkas.

Nyckelord

artificell dissipation, energiuppskattning, konservativ formulering

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