FOI-R--0676--SE
December 2002
ISSN 1650-1942
Scientific report

Mats Bengtsson

A Non-Isotropic Potts Model for Resource Allocation

Command and Control Systems
SE-581 11 Linköping

SWEDISH DEFENCE RESEARCH AGENCY

Command and Control Systems
P.O. Box 1165
SE-581 11 Linköping

FOI-R--0676--SE

December 2002

ISSN 1650-1942

Scientific report

Mats Bengtsson

A Non-Isotropic Potts Model for Resource Allocation

Issuing organization	Report number, ISRN	Report type
FOI – Swedish Defence Research Agency Command and Control Systems P.O. Box 1165 SE-581 11 Linköping	FOI-R--0676--SE	Scientific report
	Research area code 4. C4ISR	
	Month year December 2002	**Project no.** E7037
	Customers code 5. Commissioned Research	
	Sub area code 41 C4I	

Author/s (editor/s)	Project manager
Mats Bengtsson	Per Svensson
	Approved by
	Sponsoring agency
	Scientifically and technically responsible Mats Bengtsson

Report title

A Non-Isotropic Potts Model for Resource Allocation

Abstract (not more than 200 words)

An allocation problem consisting of a number of dynamic objects to be placed on a number of fixed positions so that an a priori known quantity is optimized, is solved using a Potts model with anisotropic interactions. A mean field approximation is used to reduce simulation times.

Keywords

Simulated annealing, clustering, Potts spins, resource allocation, mean field theory, sensor management

Further bibliographic information	Language English
ISSN 1650-1942	**Pages** 20 p.
	Price acc. to pricelist

Utgivare	Rapportnummer, ISRN	Klassificering
Totalförsvarets Forskningsinstitut - FOI Ledningssystem Box 1165 581 11 Linköping	FOI-R--0676--SE	Vetenskaplig rapport
	Forskningsområde 4. Spaning och ledning	
	Månad, år December 2002	**Projektnummer** E7037
	Verksamhetsgren 5. Uppdragsfinansierad verksamhet	
	Delområde 41 Ledning med samband och telekom och IT-system	

Författare/redaktör	Projektledare
Mats Bengtsson	Per Svensson
	Godkänd av
	Uppdragsgivare/kundbeteckning
	Tekniskt och/eller vetenskapligt ansvarig Mats Bengtsson

Rapportens titel (i översättning)

En icke-isotropisk Pottsmodell för resursallokering

Sammanfattning (högst 200 ord)

En Pottsmodell med en icke-isotropisk växelverkan används för att lösa problemet med att allokera ett antal dynamiska objekt till ett antal fasta positioner. Allokeringen sker på ett sätt, som optimerar en till problemet given funktion. En medelfältsapproximation används för att reducera beräkningstiderna. Metoden kan exempelvis användas i avancerade sensortillämpningar.

Nyckelord

Simulerad termalisering, klustring, Pottsspinn, resursallokering, medelfältsteori, sensorhantering

Övriga bibliografiska uppgifter	Språk Engelska
ISSN 1650-1942	**Antal sidor:** 20 s.
Distribution enligt missiv	**Pris: Enligt prislista**

Contents

1 Motivations and Applications

With the development of complex and low-cost microelectronics, the number of sensors used in defense and civilian systems proliferate. The large data sizes to be processed has pushed the analyzing problem far beyond what can be handled by a human operator. This motivates an emerging interest in research of automatic or semi-automatic management of sensor resources for improving overall performance beyond fusion of data.

Sensor management is the process whereby sensor resources are allocated to optimize some kind of yield, and is intimately related with optimization of the information fusion process. Sensor resources can be any features of a sensor, such as position, direction, area of interest, operation modes, scheduling, etc. The purpose of sensor management is to optimize the data fusion performance by providing some kind of feedback. This performance index must be a quantifiable observable, denoted *figure of merit* in [1]. Its definition is problem dependent, and a description is limited to a specific technical system. In our analysis we assume that this function (*figure of merit*) can be described in terms of a quadratic function involving an abstraction of the sensor attributes as the dynamic quantities, and where the specific problem is encoded as a (static) interaction matrix of the sensor attributes. A concrete case may be a situation where the sensor attributes are (discrete) geographical positions, and the interaction matrix encodes some kind of gain describing the outcome of each possible configuration for placing sensors at the positions.

This is a combinatorial problem, and an exhaustive search has a computing complexity that raises exponentially with the problem size. A specific problem, however, may be solvable with a specifically designed non-exhaustive optimization algorithm. Our task here is to point to a method that has a more general applicability, albeit using some assumptions. In physics terms, the energy function is quadratic, which means that the interaction between any two dynamical objects is linear. In situations where this description is not valid on an actual real world problem, our method can be seen as a linearization, which may be used as a starting point for a refined analysis. Another sacrifice is that we assume the state space for each object (sensor) to be identical.

2 The Model

We will consider the following problem in this paper. Assume that there are K identical objects to be placed on N different positions, so that a

certain quantity is optimized. Any object may, of course, only be placed in a single position at a time, but a single position may contain zero or more objects. We also assume that $N > K$. We denote the objects with indices $i, j, \ldots : 1 \leq i, j \leq K$, and positions with $a, b, \ldots : 1 \leq a, b \leq N$. The dynamic quantities, interpreted as Potts spins, are $S_{ia} = 0, 1$, where $S_{ia} = 1$ means that object i is in position a.

We define an energy function,

$$E = -\frac{1}{2} \sum_{i,j} \sum_{a,b} D_{ab} S_{ia} S_{jb}, \tag{1}$$

with $0 \leq D_{ab} \leq 1$ being a priori known constants. Pick two objects i^* and j^*, and two positions a^* and b^*, then $(D_{a^*b^*} + D_{b^*a^*})/2$, or $D_{a^*b^*}$ if symmetric interactions, is precisely the cost (or energy) of this particular configuration.

In comparison with energy functions ordinarily used in Potts models for computation tasks, this is a peculiar model since the interactions depend on the spin degree-of-freedom and not on the spin sites. The interactions are therefore homogeneous, but non-isotropic.

For a description and analysis of ordinary isotropic Potts models used for clustering and optimization, see for instance [2],[3],[4].

The task is now to minimize the energy function in order to reach a solution of the problem. In other words, find the configuration of the state described by $\{S_{ia}\}$ that corresponds to minimum of E. This is obviously a combinatorial optimization problem that suffers from the usual combinatorial explosion of the size of the state space. We therefore seek a dynamic solution that may find at least an approximate solution in a polynomial amount of time.

The energy function is nonlinear which complicates things. Furthermore, it is likely that the energy function is full of local minima if described in terms of some continuous state space. This is the reason simulated annealing is used, since it is much better in this respect than many other search methods. However, Monte Carlo simulations are usually very time consuming. Therefore a mean field approximation is used that to some extent resembles a Monte Carlo simulation, but that avoids the excessive time consumption.

We start by rewriting the energy functions by picking out a particular object i^*, and linearizing the energy function with respect to this spin.

$$E = -\frac{1}{2} \sum_{a,b} D_{ab} \left[\sum_{i,j} S_{ia} S_{jb} \right] = \tag{2}$$

$$-\frac{1}{2} \sum_{a,b} D_{ab} \left[S_{i^*a} S_{i^*b} + S_{i^*a} \sum_{j \neq i^*} S_{jb} + S_{i^*b} \sum_{j \neq i^*} S_{ja} \right] + T[\{\hat{S}\}], \tag{3}$$

where $T[\{\hat{S}\}]$ does not depend on the particular spin i^*. Define fields

$$H_{ia}[S] = \sum_{j \neq i} S_{ja}, \tag{4}$$

which represent the total number of objects, not counting object i, at position a. Eq. (3) is then written as

$$E = -\frac{1}{2} \sum_{a,b} D_{ab} \left[S_{i^*a} S_{i^*b} + S_{i^*a} H_{i^*b} + S_{i^*b} H_{i^*a} \right] + T[\{\hat{S}\}]. \tag{5}$$

A short note: the first term in eq. (5) can be further simplified, from $\sum_{a,b} D_{ab} S_{i^*a} S_{i^*b}$ to $\sum_a D_{aa} S_{i^*a}$. But this will not be true in a mean field representation where the discrete S variables are replaced by continuous valued variables. Although the ground state is kept unchanged by using this simplification, the dynamics that leads to the ground state will be different. We therefore keep it in its original form.

Derivation of the mean field equations of motion is straightforward. We seek the thermal average of S_{i^*a}, $\langle S_{i^*a} \rangle$, expressed directly as:

$$\langle S_{i^*a} \rangle = \frac{1}{Z} \sum_{\{S\}} S_{i^*a} \, e^{-\beta E[S]} = \frac{1}{Z} \sum_{\{\hat{S}\}} \sum_{S_{i^*}} S_{i^*a} \, e^{-\beta E[S]} \tag{6}$$

where the partition sum $\sum_{\{S\}}$ is over each *legal* state (fulfilling the constraint) of the spin variables. The trick is to introduce a common factor $\sum_{S_{i^*}} \left(e^{-\beta E[S]} \right)$. Eq. (6) then becomes:

$$\langle S_{i^*a} \rangle = \frac{1}{Z} \sum_{\{\hat{S}\}} \left[\sum_{S_{i^*}} \left(e^{-\beta E[S]} \right) \frac{\sum_{S_{i^*}} \left(S_{i^*a} \, e^{-\beta E[S]} \right)}{\sum_{S_{i^*}} \left(e^{-\beta E[S]} \right)} \right] \tag{7}$$

Noting that the common $T[\{\hat{S}\}]$ term can be canceled in the ratio, we get for the ratio

$$\frac{\sum_{S_{i^*}} S_{i^*a} \exp\left(\frac{\beta}{2} \sum_{b,c} D_{bc} \left(S_{i^*b} S_{i^*c} + S_{i^*b} H_{i^*c} + S_{i^*c} H_{i^*b} \right) \right)}{\sum_{S_{i^*}} \exp\left(\frac{\beta}{2} \sum_{b,c} D_{bc} \left(S_{i^*b} S_{i^*c} + S_{i^*b} H_{i^*c} + S_{i^*c} H_{i^*b} \right) \right)} \tag{8}$$

Sum over each legal state; $\{S\} = \{(1000\ldots), (0100\ldots), (0010\ldots), \ldots\}$. Do this for both the numerator and denominator, to get the ratio:

$$\frac{\exp\left(-\frac{\beta}{2} \left[D_{aa} + \sum_c (D_{ac} + D_{ca}) H_{i^*c} \right] \right)}{\sum_b \exp\left(-\frac{\beta}{2} \left[D_{bb} + \sum_c (D_{bc} + D_{cb}) H_{i^*c} \right] \right)} \tag{9}$$

Plug this into eq. (7), and we get

$$\langle S_{i^*a} \rangle = \left\langle \frac{\exp\left(-\frac{\beta}{2}\left[D_{aa} + \sum_c (D_{ac} + D_{ca}) H_{i^*c}\right]\right)}{\sum_b \exp\left(-\frac{\beta}{2}\left[D_{bb} + \sum_c (D_{bc} + D_{cb}) H_{i^*c}\right]\right)} \right\rangle \tag{10}$$

where the definition of stochastic averages have been used.

Mean field theory implies $\langle f[S] \rangle = f(\langle S \rangle)$ for any stochastic variable S. Defining $V_{ia} = \langle S_{ia} \rangle$, noting that everything so far is valid for any object i, and not only the specifically picked one i^*, we get the final mean field equations:

$$V_{ia} = \frac{\exp\left(\frac{\beta}{2}\left[D_{aa} + \sum_c (D_{ac} + D_{ca}) H_{ic}\right]\right)}{\sum_b \exp\left(\frac{\beta}{2}\left[D_{bb} + \sum_c (D_{bc} + D_{cb}) H_{ic}\right]\right)}, \tag{11}$$

but with,

$$H_{ic} = \sum_{j \neq i} V_{jc}. \tag{12}$$

If interactions are symmetric, as they are in any physical system, eq. (12) is further simplified to

$$V_{ia} = \frac{\exp\left(\beta\left[\frac{1}{2} D_{aa} + \sum_c D_{ac} H_{ic}\right]\right)}{\sum_b \exp\left(\beta\left[\frac{1}{2} D_{bb} + \sum_c D_{bc} H_{ic}\right]\right)}. \tag{13}$$

3 Trivial Example

In a situation where the interaction matrix D_{ab} describes metric distances in a room, the problem has a simple geometric interpretation[1]. The minimum of the energy function eq. (1) is identical to the maximum of $\sum_{i,j} \sum_{a,b} D_{ab} S_{ia} S_{jb}$, which is just the sum of all pairwise distances between all objects.

If we define

$$(a^*, b^*) = \arg \max_{a,b} D_{ab} \tag{14}$$

as the two most extreme positions, these are candidate positions for the objects. For two objects ($N = 2$) they are also the optimal positions. For more than two objects, they are at least populated, but additional positions that are close to the circumference described by a^* and b^* may also be populated.

[1] It is not unlikely that it is possible to find an efficient algorithm that solves this special case.

4 Linear Term

The flexibility of the Potts model formulation of the problem allows other cost terms for adapting to a specific optimization task. As an example, a linear term can be added in order to describe the cost of having an object at position a

$$E = \sum_i \sum_a C_a S_{ia}, \tag{15}$$

with $C_a \geq 0$. Having this term alone gives trivial solutions, but combined with eq. (1) more complex problems can be described.

It is also possible to have a term that describes the cost of a particular object i at position a as

$$E = \sum_i \sum_a C_{ia} S_{ia}. \tag{16}$$

5 High Temperature Expansion

Since this model is described in terms of mean field theory and statistical mechanics, it also shares phenomena known from physical systems. One such phenomenon that is also valuable for practical simulations of the system is the critical temperature of the first phase transition. At a high temperature, the system is in its completely symmetric state where all positions are equally populated by all objects. In terms of the mean field state variables, $V_{ia} = 1/N \ \forall \ i \ \& \ a$.

For an isotropic Potts spin model, where interactions are indifferent to the spin direction, this high-temperature fixpoint is a fixpoint for all temperatures. However, it is only for temperatures above the critical temperature it is a stable fixpoint. At temperatures below the critical one, the fixpoint is unstable.

The behavior of the non-isotropic Potts model is different. If the symmetric solution $V_{ia} = 1/N$ is plugged into eq. (13), assuming $D_{aa} = 0$, we get $H_{ia} = (K-1)/N$, and the left hand side of eq. (13) is $1/N$, while the right hand side is

$$\frac{\exp\left(\beta \frac{K-1}{N} \sum_c D_{ac}\right)}{\sum_b \exp\left(\beta \frac{K-1}{N} \sum_c D_{bc}\right)}, \tag{17}$$

which are clearly different. Only asymptotically $\beta \to 0$ this is a solution. Thus, the phase transition for the non-isotropic Potts model is different from the ordinary isotropic Potts model. The transition seems "smoother" without an abrupt state change at any critical temperature. The situation is similar to that of a first order phase transition versus a second order phase transition.

Assuming that $D_{ab} = D_{ba}$ and $D_{aa} = 0 \quad \forall \quad a \ \& \ b$, we make a series expansion of V_{ia} for small β (high temperatures). The equation of motion is

$$V_{ia} = \frac{\exp\left(\beta \sum_c D_{ac} H_{ic}\right)}{\sum_b \exp\left(\beta \sum_c D_{bc} H_{ic}\right)}, \tag{18}$$

with $H_{ia} = \sum_{j \neq i} V_{ja}$. Make the following ansatz:

$$V_{ia} = V_{ia}^{(0)} + \beta V_{ia}^{(1)} + \frac{\beta}{2} V_{ia}^{(2)} + O(\beta^3) \tag{19}$$

which is just plugged into eq. (18). Expanding the exponential factors systematically in terms of β, it is just to identify the factors of each power of β to be able to find the V_{ia} expansion in terms of the interaction matrix. After some straightforward but tedious algebra we get

$$V_{ia}^{(0)} = \frac{1}{N} \tag{20}$$

$$V_{ia}^{(1)} = \frac{N-1}{N} \left(D_{a\bullet} - \frac{1}{N} D_{\bullet\bullet} \right) \tag{21}$$

$$V_{ia}^{(2)} = \frac{(N-1)^2}{N^3} \left(D_{a\bullet} D_{a\bullet} + 2 \sum_c D_{ac} D_{c\bullet} - \frac{4}{N} D_{a\bullet} D_{\bullet\bullet} - \frac{2}{N} \sum_c D_{c\bullet} D_{c\bullet} - \frac{N-4}{N^2} D_{\bullet\bullet} D_{\bullet\bullet} \right) \tag{22}$$

where

$$D_{a\bullet} = \sum_b D_{ab} \tag{23}$$

$$D_{\bullet\bullet} = \sum_{a,b} D_{ab}. \tag{24}$$

This enables us to get an analytical expression for V_{ia} to use as a starting point in the simulation.

6 Dynamics

6.1 Bifurcations

Consider two positions a distance d apart, and two objects. This problem has a trivial solution, but is anyway useful as it clearly illustrates a very

naughty dynamic issue, that of bifurcations. The distance matrix is

$$D = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}, \tag{25}$$

and the low temperature equations ($\beta \to \infty$) gives $V_{ia} = 1$ for

$$\arg_a \max \sum_c D_{ac} \sum_{j \neq i} V_{jc}, \tag{26}$$

and $V_{ia} = 0$ else. If we interpret the variables V_{ia} and D_{ab} in terms of matrices, with first index row index and second index column index as usual, the update equation is just a simple matrix multiplication $M \times D$, with $M_{ia} \equiv \sum_{j \neq i} V_{ja} = \sum_j V_{ja} - V_{ia}$. This assumes $D_{ab} = D_{ba}$. For each row (that is an object) the column index corresponding to the maximum value "wins", and $V_{ia} = 1$.

If we start out with the correct solution,

$$V_{ia} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{27}$$

it also stays there, as expected. But if we start out with a situation with both objects in the same position,

$$V_{ia} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \tag{28}$$

it switches to

$$V_{ia} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{29}$$

after a single iteration. After one further iteration, it switches back again. This is a typical bifurcating dynamics, and is highly undesirable. The cause of this behavior is due to the batch (parallel) updating scheme. It is possible to interpret it the following way: both objects repel each other when they are in the same position, and both therefore want to move to the other position, simultaneously since parallel updating takes place synchronously. The cure for this is to update the system serially, that is, one at a time. This removes the problem completely.

6.2 Increasing Energy

There exist a peculiar phenomenon where the energy actually increases during the annealing process, despite our previous claims that the task is to find

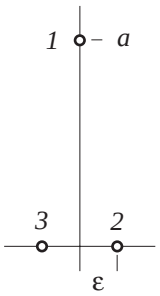

Figure 1: *Three positions, 1, 2, 3, at coordinates* $(0, a)$, $(\epsilon, 0)$ $(-\epsilon, 0)$, *respectively.*

the absolute minimum of the energy function. However, this turns out to be an artifact of the mean field model.

Consider a situation with three positions and two objects, illustrated in Fig. (1). Position 2 and 3 are very close. This is a degenerate case since one of the objects may end up in either position 2 or 3 with identical energies $E = -\sqrt{a^2 + \epsilon^2}$. During the annealing process, one object moves to position 1, while the other takes up an intermediate position, just between 2 and 3, before it breaks the degeneracy, and moves to either one of them. The energy for the intermediate state is $E = -\sqrt{a^2 + \epsilon^2} - \epsilon/2$, actually lower than the ground state!

7 Simulation Results

Some numerical simulations and benchmarking has been performed. Fig. (2) shows an annealing process with $N = 20$ positions and $K = 8$ objects. Each object traces out a line as it moves in this 2D space. As can be seen the extreme positions are the optimal ones, and they may be populated by one or many objects.

For the identical case, the actual state of one of the mean field spin state variables V_{ia} is plotted at the top in Fig. (3)

As an illustration of the peculiar phenomenon with increasing energy, discussed above, Fig. (4) shows this clearly.

A more complex situation with $N = 80$ positions is shown in Fig. (5).

In order to better verify that the results are truly optimal, an exhaustive search algorithm is used that tries every combination possible for finding the true optimum. For large scale problems, where problems, if any, will show up, it is impossible to verify results with exhaustive search, since cpu

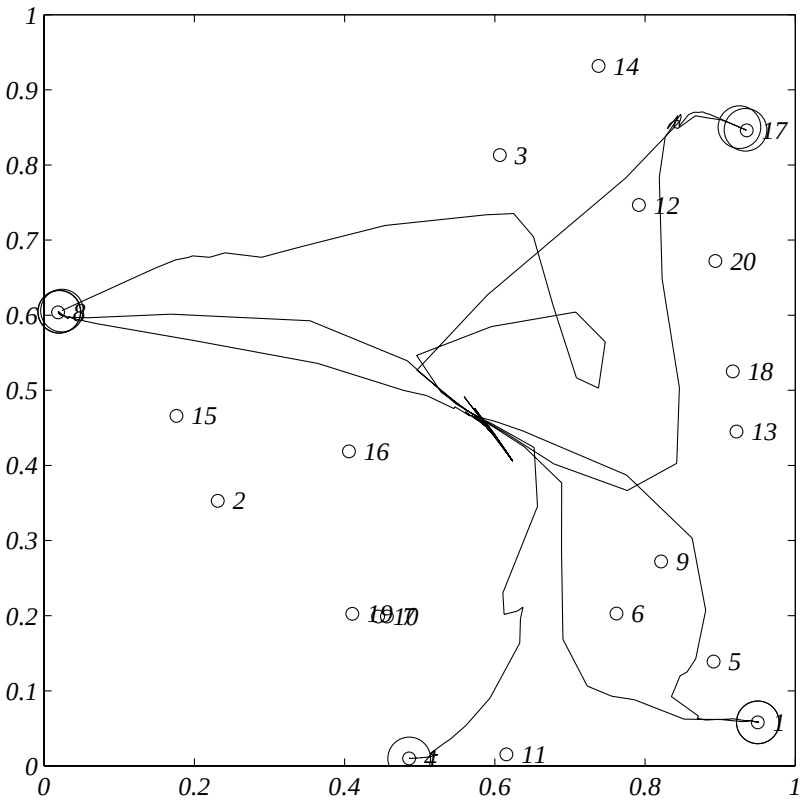

Figure 2: *An annealing process with $N = 20$ and $K = 8$, where positions are small circles and objects are larger circles. Each object traces out a line as it moves from its original position in the center of gravity of the positions, to its final position.*

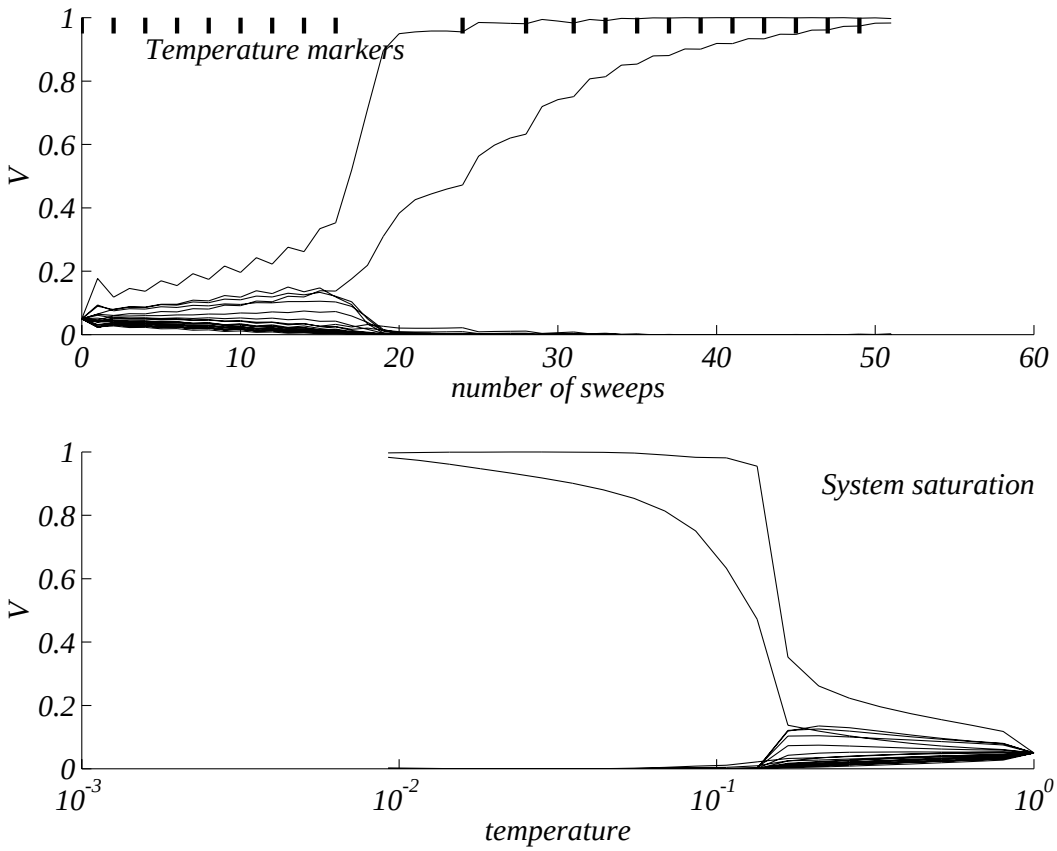

Figure 3: *Top: The state of one of the spin variables V_{ia} and all its component versus the number of sweeps. As a comparison, the temperature markers as the temperature is decreased are shown at the top. Bottom: Same as top, but plotted versus the temperature. The top line is, however, the total system saturation.*

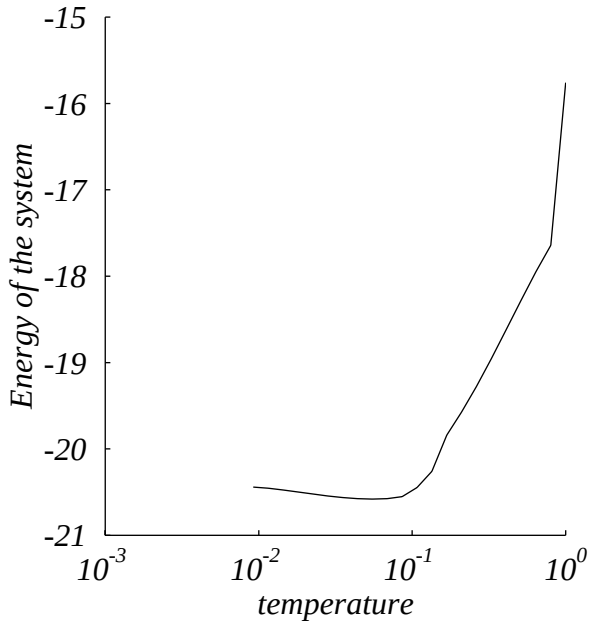

Figure 4: *The total energy of the system plotted versus the temperature.*

times are excessive. Running many cases with $N = 20$ and $K = 8$, all results are optimal, but the exhaustive search algorithm (implemented in C) is about 20 times slower than the Potts model (implemented in Matlab). If both algorithms were implemented using the same languages, the computing complexity would be approximately another factor of ten in favor of the Potts model. About 20 runs were also made with $N = 40$ and $K = 8$, where all results from the Potts model and the exhaustive search algorithm agreed. The median cpu time for the Potts model was 0.15 seconds. The exhaustive algorithm takes typically five cpu minutes for the identical problems.

A somewhat more detailed computing complexity analysis is given in Fig. (6) and Fig. (7). For varying N and fixed K, cpu time raises quadratically with N, but with fixed N and varying K, the cpu time is close to linear. This gives a first indication of a complexity of $O(N^2K)$.

Acknowledgments: Thanks to Johan Schubert for introducing me to the problem, and for pointing out a possible Potts solution.

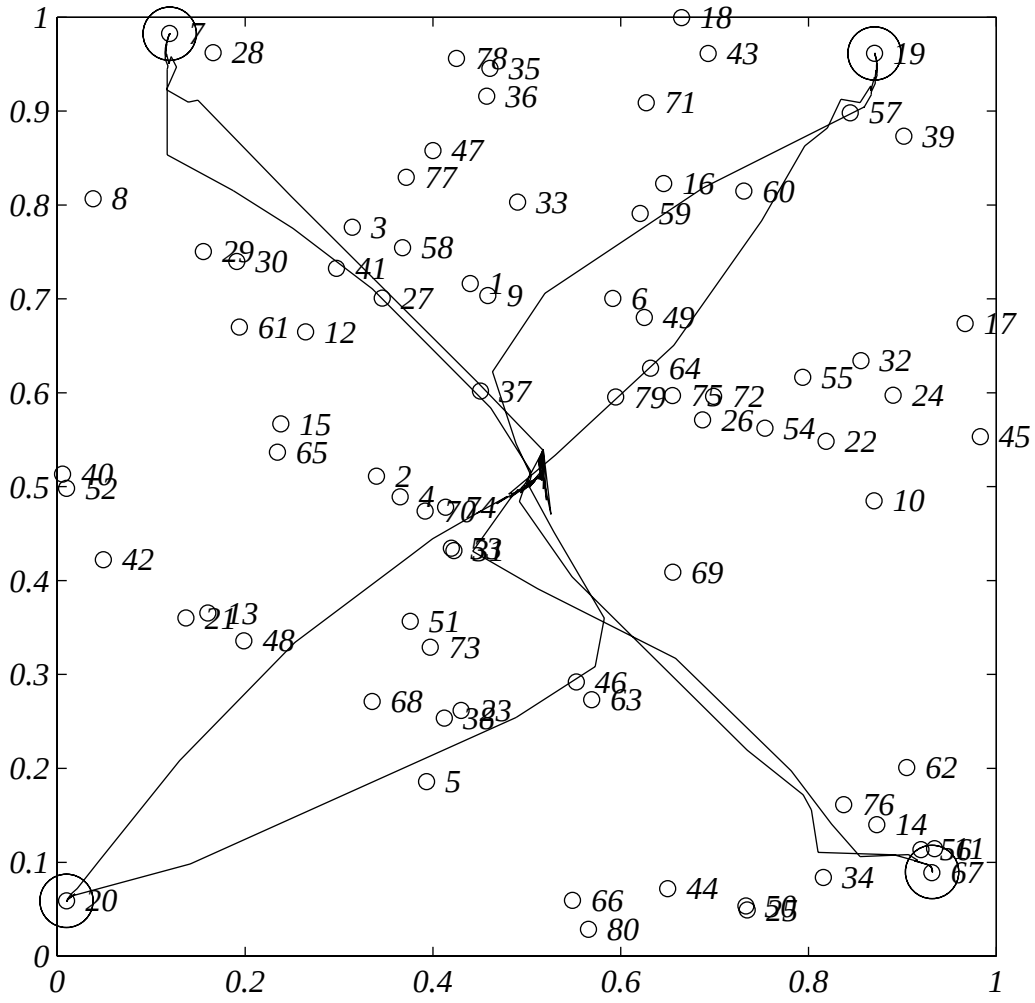

Figure 5: *An annealing process with $N = 80$ and $K = 8$, where positions are small circles and objects are larger circles. Each object traces out a line as it moves from its original position in the center of gravity of the positions, to its final position.*

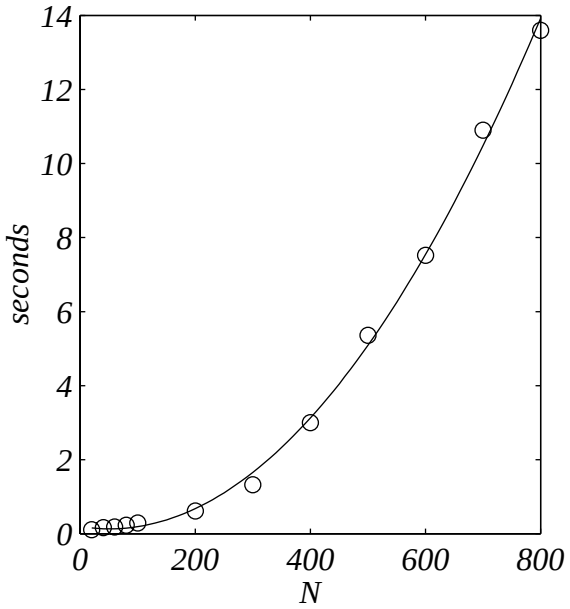

Figure 6: *Cpu time (for a Matlab implementation) for the case with fixed* $K = 8$ *and varying* N*. The line is an interpolated second order polynomial.*

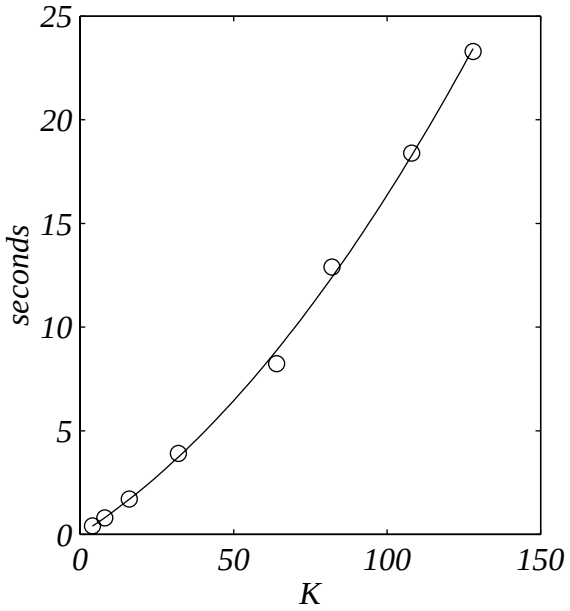

Figure 7: *Cpu time (for a Matlab implementation) for the case with fixed* $N = 256$ *and varying* K*. The line is an interpolated second order polynomial.*

References

[1] Blackman, S. and Popoli, R. Sensor management, In: Design and analysis of modern tracking systems, Artech House, Norwood, MA, pp. 967-1068 (1999).

[2] M. Bengtsson and P. Roivainen, "Using the Potts Glass for Solving the Clustering Problem," *Int. J. Neural Syst.* **6**, 119-132 (1995).

[3] M. Bengtsson, "Graph Optimization and the Potts Spin Model," Defence Research Establishment report, FOA-R--98-00860-616--SE, (1998).

[4] M. Bengtsson and J. Schubert, "Dempster-Shafer Clustering using Potts Spin Mean Field Theory," *Soft Computing* **5**, 215-228 (2001).