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Parameter Uncertainty in an Air Defence Missile Model:

A Sensitivity Analysis Study

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Parameter Uncertainty in an Air Defence Missile Model:A Sensitivity Analysis Study

Abstract

This study was conducted within the project Technical Threat Systems Analysis. The main purpose of this study has been to make a preliminary investigation of potential methods based on Sensitivity Analysis (SA), in order to be a support in the process of model development, analysis and use. SA is used to increase the confidence in the model and its predictions and provide a powerful tool in an integrated approach for model development and performance evaluation.

In this report we emphasise methods that have potential to analyse the sensitivity of dynamical systems and in order to evaluate the proposed SA methods a case study was defined. The chosen case for this study was a simplified model of a 3-dimensional (3-D) Surface to Air Missile (SAM) against an aerial target. In order to define the case to be studied by different SA methods, a set of model parameters and output signals were selected and studied.

Local and global SA methods have been applied on the SAM model and the overall conclusion of this study was that SA has a strong potential to be a useful tool in evaluating a model in terms of how parametric uncertainties affect the output of missile analysis systems. Results from these SA methods are presented in this report.

Keywords

Sensitivity analysis, missile model, modelling, simulation

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Rapportens titel

Parameterosäkerhet av en luftvärnsrobotmodell:En känslighetsanalysstudie

Sammanfattning

Denna studie har utförts inom projektet Teknisk Hotsystemanalys. Huvudsyftet med denna studie har varit att preliminärt undersöka möjliga metoder baserade på känslighetsanalys, i syfte att stödja modellutveckling, analys och användning. Känslighetsanalys kan användas för att öka en modells tillförlitlighet och prediktering samt att vara ett kraftfullt integrerat hjälpmedel för modellutveckling och utförandeutvärdering.
I denna rapport betonas metoder som har potential att analysera känsligheten av dynamiska system. I syfte att utvärdera de föreslagna metoderna inom känslighetsanalys har ett fall definierats. I denna studie har en förenklad modell av en tredimensionell (3-D) luftvärnsrobotmodell valts. För att definiera det med känslighetsanalys studerade fallet, har modellparametrar och utsignaler studerats.
Lokala och globala känslighetsanalysmetoder har tillämpats på luftvärnsrobotmodellen och den övergripande slutsatsen av studien var att känslighetsanalys har en stor potential att vara ett användbart verktyg för att utvärdera modeller vad gäller parametrisk osäkerhetspåverkan av ett robotsystems utsignaler. Resultat från dessa känslighetsanalyser presenteras i denna rapport.

Nyckelord

Känslighetsanalys, robotmodell, modellering, simulering

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Contents

1. Introduction

1.1 Background

As the capability of computer simulation increases, modelling and simulation is used widely in the investigation of complex physical systems and decision making. The international defence community is increasingly being asked to accept more evidence from simulations and less from traditional live testing.

To allow this, the simulation results have to be credible. Credibility is based on confidence in the correctness of the simulation model and its appropriateness to the application of interest. The military services need support to assess the credibility of the models involved in their process of materiel acquisition and integration.

The main purpose of this study is to investigate potential methods based on Sensitivity Analysis (SA) to assist modelers throughout the process of model development, analysis and use.

Sensitivity Analysis (SA) is a body of scientific methods for studying the relationships between information flowing in and out of a model. Originally, SA was a method aimed at measuring the impact of changes or uncertainties in the input variables and parameters on model outputs. The methods have since then been extended to incorporate model conceptual uncertainty, i.e. uncertainty in model structures, assumptions and specifications.

As a whole, SA is used to increase the confidence in the model and its predictions and provide a powerful tool in an integrated approach for model development and performance evaluation.

SA is close related to Uncertainty Analysis (UA). Although closely related, they address different problems. Uncertainty Analysis assesses the uncertainty in model outputs due to input uncertainties while sensitivity analysis assesses the contributions of the inputs (not uncertainties) to the total uncertainty in the outcomes.

In this report we emphasise methods that have potential to analyse the sensitivity of dynamical systems, e.g. aircrafts and missiles.

1.2 Computer simulation models

Modelling and simulation is a process of building, analysing and using theoretical and experimental results in order to summarise a body of knowledge, to make predictions or to understand system dynamics. The stage of building consists of defining the problem and the system, collecting knowledge (processes, parameters and direct observations), developing a model concept, translating it to a mathematical model and converting this into a computer program. A mathematical model is defined by a series of equations, input factors, parameters, and variables aimed to characterise the process being investigated. Input is subject to many sources of uncertainty, including errors of measurement, absence of information and poor or partial understanding of the driving forces and mechanisms. This imposes a limit on our confidence in the response or output of the model.

Many of the models are also structured hierarchically, i.e. system of systems. This means that a model can be decomposed into interacting sub-models, which again can be decomposed into interacting sub-models, until an atomic level with no further decomposition. These sub-models typically contain parameters, and the model output can be highly sensitive to small changes in the parameter values. Insufficient and imprecise knowledge about the models inputs, parameters and structures, imposes a limit on our confidence in the response or output of the model. The usefulness of the models is critically depended on how accurately it can represent the aspects of the reality we want to study.

1.3 Model Verification, Validation, and Accreditation (VV&A)

The Department of Defense (DoD) in USA and the military services have prepared directives and guidelines to provide general instructions on how, when, and under what circumstances, formal VV&A procedures should be employed to establish the credibility of the models and simulations. The purpose of VV&A is to assure development of correct and valid simulations and to provide simulation users with sufficient information to determine if the simulation can meet their needs. VV&A incorporates three distinct processes that gather and evaluate evidence to determine, based on the simulations intended use, the simulations capabilities, limitations, and performance relative to the real-world. The formal definitions for these processes are given below [1]:

- *Verification* – the process of determining that a model implementation and its associated data, accurately represent the developer's conceptual description and specifications.
- *Validation* – the process of determining the degree to which a model and its associated data provide an accurate representation of the real world from the perspective of the intended use of the model.
- *Accreditation* – the official certification that a model, simulation, or federation of models and simulations and its associated data is acceptable for use for a specific purpose.

SA provides methods for performing validation of a model possibly assessing the uncertainties associated with the modelling process and with the outcome of the model itself.

1.4 Modelling problems addressed by SA

Sensitivity analysis is used throughout the development process not just as an analysis tool but as a development, integration, test, verification and sustainment resource. In this way, SA also touches on the difficult problem of model quality. Modelers may conduct SA to perform:

- *Model identification*: Model identification aims to find out if a model is appropriate for the available data. SA can identify the most appropriate model structures and specifications competing to describe available evidence.
- *Calibration*: Calibration can be used to find the best match between the model and direct system observations. During or subsequent to calibration, the remaining uncertainty in the model outcomes has to be estimated in an uncertainty analysis.
- *Parameter tuning*: Evaluating the magnitude of the effects of parameter uncertainty on the model performance and guiding a model refinement process to assure realistic behavior.
- *Model improvement*: SA can be used to guide the model refinement in the regions of the most significant factors or sub-systems and increase their level of accuracy.

- *Model reduction*: Different uncertainties in the input factors or sub-systems impact differently on the reliability, the robustness and the efficiency of the model. SA can be used to identify and eliminate those factors or sub-systems with little or no impact at the model performance. In this way irrelevant parts of the model can be dropped.

- *Validation*: SA can be used to provide acceptance criteria to decision-makers by quantifying the degree of detail that must be present for appropriate results and the degree of correctness needed in correspondence to the real systems response.

1.5 Developing missile and aircraft models at FOI

Objectives of the project Technical Threat Systems Analysis are to provide our customers within armed Forces and other military services with sustainable knowledge in the field of missiles and military aircrafts. Typically, our knowledge is summarised in a form of mathematical and computer simulated models aimed to study possible outcomes in a variety of technical, tactical and electronic warfare scenarios. These models provide a powerful instrument to explore system performance, make predictions and answer "what if" questions. For instance, simulations of a missile to aircraft duel scenario can provide valuable information about how electronic countermeasures are best combined with tactical maneuvers for escape.

The underlying data to the models is often an assembly of many pieces of information, of varying quality and from disparate sources. Generally the information available is incomplete, uncertain and at times erroneous. Some of the available information is explicitly given as aerodynamic tables, physical and mechanical data and readings from experimental trails. But much of the information is implicitly assumed by the modeler, deduced from physics and as a last resort created by a mental process of reverse engineering of similar systems. The models must be fit for the purpose for which they are created, and different purposes may require different levels of complexity. Consequently some parts of a model need to be encoded in greater detail while others with little influence on the purpose are kept in a summary or omitted.

The uncertainties mentioned above impose a limit on the confidence of the models and the main questions underlying this work are:

- Can we quantify an overall uncertainty bound to the answers produced a model? Can we identify those parts of the model that needs to be improved?

- What factors affect model quality the most? How can we be sure that a model will be valid and fit for its purpose when models in general are a complex assembly of modelling assumptions and uncertainties?

- How can we be sure that a model is used to answer the right questions? What kind of questions can we pose on models?

- How can we reduce the complexity to comply with the degree of accuracy required by the questions being addressed? What factors have only a minor effect on the reliability, robustness and efficiency of a model?

There is a large body of literature addressing many of the above questions theoretically and experimentally, providing knowledge and experience of practicing sensitivity and uncertainty analysis in many fields. A common starting point is the book by Saltelli [10]. Some key papers describing application of SA on specific topics and containing many valuable pointers are: performance assessment in radioactive waste disposal [5], complex kinetic systems [12], traffic operation and management [9] and incompressible aeroelastic systems [3]. SA has also influenced many other scientific fields to improve their algorithms such as neural networks [4], and optimization techniques [6].

1.6 Outlook

The first chapter gives a brief introduction into two commonly used methods in SA. Next chapter describes a simplified model of a missile to aircraft duel scenario. This is a much simplified model compared to the regular models and it is used here as an evaluation case of the SA methods. Then follows a chapter describing the simulation settings and the obtained results. Finally, the last chapter discusses the ability of SA to address the posed questions and identifies some directions for future work.

2. Methods

2.1 Sensitivity analysis methods

The general question that can be answered by sensitivity analysis is: How do changes in the model parameters affect the output of the system?

We will study systems that can be described by stationary or time dependent functions that depend on a constant parameter vector

$$\boldsymbol{y}(t, \boldsymbol{p}), \quad \boldsymbol{p} \in \mathcal{P} \tag{2.1}$$

where the vector valued function $\boldsymbol{y}(t, \boldsymbol{p})$ takes values in $\mathcal{R}^n$ and the parameter vector $\boldsymbol{p}$ belongs to a subset $\mathcal{P}$ of $\mathcal{R}^m$. We will study how the function $\boldsymbol{y}(t, \boldsymbol{p})$ varies when the parameter vector $\boldsymbol{p}$ is varied.

For our considerations, we will assume that the studied function $\boldsymbol{y}(t, \boldsymbol{p})$ is defined through a set of ordinary differential equations, together with an output functional, depending on the state $\boldsymbol{x}$

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{p}), \quad \boldsymbol{x}(t_0, \boldsymbol{p}) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t, \boldsymbol{p}) &= \boldsymbol{h}(t, \boldsymbol{x}(\tau, \boldsymbol{p})), \quad \tau \in [t_0, t] \end{aligned} \tag{2.2}$$

Note that the output functional depends on the whole time range from t_0 up to the present time t. Care should be taken when choosing the output functional $\boldsymbol{h}$, different choices will measure different aspects of the underlying system. One example of an output functional is the usual output function for a linear dynamical system

$$\boldsymbol{y}(t, \boldsymbol{p}) = \boldsymbol{C}\boldsymbol{x}(t, \boldsymbol{p}) \tag{2.3}$$

where $\boldsymbol{C}$ is a constant matrix. Another example is a measure of the average difference between a given trajectory $\boldsymbol{x}(\tau, \boldsymbol{p})$ and a nominal trajectory $\hat{\boldsymbol{x}}(\tau, \hat{\boldsymbol{p}})$

$$y(t, \boldsymbol{p}) = \frac{1}{t} \int_0^t \| \boldsymbol{x}(\tau, \boldsymbol{p}) - \hat{\boldsymbol{x}}(\tau, \hat{\boldsymbol{p}}) \| \, \mathrm{d}\tau \tag{2.4}$$

Sensitivity analysis methods can be divided into local and global analysis methods, although this division is somewhat arbitrary, see [10]. Local sensitivity analysis is related to a single point $\boldsymbol{p}_0$ in the parameter set $\mathcal{P}$, and is often based on approximations of $\boldsymbol{y}(t, \boldsymbol{p})$ around $\boldsymbol{p}_0$ by Taylor-series expansions. The approximations are less accurate away from $\boldsymbol{p}_0$. In global analysis, on the other hand, one tries to capture how the output of the system behaves when the parameters are allowed to vary in the entire parameter set. Examples of global sensitivity analysis methods are sampling- and optimization-based methods. In a sampling-based sensitivity analysis method the model is executed repeatedly for combinations of values sampled from the parameter set, based upon some sort of probability distribution, and then analysis is made on the correlation between input and output data. An optimization-based sensitivity analysis method is one for which maximum and minimum departure from a nominal output value $y(t, \boldsymbol{p}_0)$ is sought, measured in some norm. The parameter vector $\boldsymbol{p}$ is allowed to vary in the entire parameter set $\mathcal{P}$.

Local analysis methods are usually simpler to carry out and are computationally less demanding than global analysis methods.

2.2 Local methods

A local sensitivity analysis method is conducted in a small region around the nominal model where a linear approximation is accurate enough. The analysis reveals the *local behavior* of the output signals, subjected to small changes of the model uncertainty parameters. The local analysis can be conducted both around a local equilibrium point in the state space or along a nominal trajectory in the state space.

In this study the main focus is on dynamical systems, which can be described as a set of time dependent first order differential equations with parametric uncertainties, as in equation (2.2). The aim of local SA is to assess the behavior of the solution of such a system, subjected to (2.2) which can be expressed by the partial derivative with respect to $\boldsymbol{p}$. This derivative can be approximated with:

$$t_{i,j} = \frac{\partial y_i(t, \boldsymbol{p})}{\partial p_j} \approx \frac{y_i(t, \boldsymbol{p} + \Delta \boldsymbol{p}_j) - y_i(t, \boldsymbol{p})}{\Delta p_j} \tag{2.5}$$

where Δp_j is a small disturbance in the j-th direction in the admissible parameter space. The entries in the sensitivity matrix $\boldsymbol{S}$ were approximated by (2.6) which was normed compared to (2.5) in order to get the *relative* sensitivity of the parameters.

$$s_{i,j} = \frac{p_j}{y_i} \cdot \frac{y_i(t, \boldsymbol{p} + \Delta \boldsymbol{p}_j) - y_i(t, \boldsymbol{p})}{\Delta p_j}, \quad y_i \neq 0, \quad \Delta p_j \neq 0 \tag{2.6}$$

If this is done for all output signals and all uncertainty parameters , the local sensitivity matrix can be defined as:

$$\boldsymbol{T}(t) = \begin{bmatrix} \frac{\partial y_1}{\partial p_1} & \cdots & \cdots & \frac{\partial y_1}{\partial p_m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial p_1} & \cdots & \cdots & \frac{\partial y_n}{\partial p_m} \end{bmatrix} \tag{2.7}$$

The sensitivity matrix $\boldsymbol{S}$ can depend explicitly on time, if the system (2.2) is also explicitly time dependent. The local sensitivity matrix can now be used for calculating the change $\Delta \boldsymbol{y}$ from the nominal solution $\hat{\boldsymbol{y}}$ caused by a disturbance in the parameter vector $\Delta \boldsymbol{p}$:

$$\boldsymbol{y} - \hat{\boldsymbol{y}} = \Delta \boldsymbol{y} \approx \boldsymbol{T}(t) \cdot \Delta \boldsymbol{p} \tag{2.8}$$

2.3 Global methods

The following different global methods are discussed:

- Sampling-based methods
- Optimization-based methods

but the emphasis will be on sampling-based methods.

2.3.1 Sampling-based methods In sampling-based methods, one takes one or more samples of the parameter vector $\boldsymbol{p}$, evaluates the function $\boldsymbol{y}(t, \boldsymbol{p})$ and then analyses the output. The method usually involve four steps:

1. Defining the parameter distributions.
2. Generating the samples from the parameter distributions.
3. Evaluating $\boldsymbol{y}(t, \boldsymbol{p})$ for all samples.
4. Analysing the results.

Compare the above list with the list in [10] , they have further divided the fourth point into two separate points.

The first step in sampling-based methods is to define the *distributions* for the input parameters. According to [10]; "Sensitivity analysis results generally depend more on the selected ranges than on the assigned distributions". In an exploratory stage of modelling and analysis of a system, or if the distributions of the input parameters are not known, a natural choice is independent uniform or normal distributions for each parameter variation. If available, it could be advantageous to use distributions based on estimates of the parameter distributions.

The second step in sampling-based methods is to decide how to *sample* from the parameter distributions. Sampling can be based on *random* sampling, *structured* sampling, or a combination of both. In random sampling K samples $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_K$ are generated from the selected distributions. The samples can be picked in a structured way, for example the end points of the intervals defining the parameter region for the uniform distributions. A motivation for this choice is that many functions obtain their maximum or minimum at the border of a region.

The third step in sampling-based methods is to *evaluate* the function $\boldsymbol{y}(t, \boldsymbol{p}_i)$ for every sample of the parameter vector $\boldsymbol{p}$ in the set $\mathcal{P}$, $i = 1, \ldots, K$.

The fourth and final step in sampling-based methods is to *analyse* the input and output data. This part involves all kind of analysis of the data e.g. computing the mean and variance of the output $\boldsymbol{y}(t, \boldsymbol{p}_i)$, or application of other statistical methods. In reference [10] the fourth point is divided into two parts, the analysis of input and output data, into two parts, the first part relating to uncertainty analysis, for example computing mean and variance. The second part relates to sensitivity analysis, for example relating how sensitive the output is to variations of individual parameters or a group of parameters.

2.3.2 Optimization-based methods Sampling-based methods can in general never guarantee the following performance:

$$\forall \;\; \boldsymbol{p} \in \mathcal{P} \;\; \boldsymbol{y}(t, \boldsymbol{p}) \in \mathcal{Y} \tag{2.9}$$

Here performance means that the output will belong to a given set if the parameter vector are allowed to vary within a given set. Optimization-based methods aim at guaranteeing the above performance and also finding the points $\boldsymbol{p}^*$ in the parameter set $\mathcal{P}$ that causes $\boldsymbol{y}(t, \boldsymbol{p})$ to depart the most from the nominal value.

Since optimization methods are designed to find extremum points, the function under study $\boldsymbol{y}(t, \boldsymbol{p})$ should not depend explicitly on time, and here we will also assume it is a scalar function. The function will be denoted by $y(\boldsymbol{p})$ in the following discussion about optimization-based methods. It should be pointed out that for general functions $y(\boldsymbol{p})$ we cannot guarantee performance with optimization-based methods either. However, it is our opinion that optimization-based methods are better suited to find extreme points. For a scalar function $y(\boldsymbol{p})$ optimization-based methods can be described as follows:

1. Define the parameter set $\mathcal{P}$
2. Find $(\boldsymbol{p}^*_{\min}, y_{\min})$ such that $y_{\min}(\boldsymbol{p}^*_{\min}) = \min_{\boldsymbol{p} \in \mathcal{P}} y(\boldsymbol{p})$
3. Find $(\boldsymbol{p}^*_{\max}, y_{\max})$ such that $y_{\max}(\boldsymbol{p}^*_{\max}) = \max_{\boldsymbol{p} \in \mathcal{P}} y(\boldsymbol{p})$

Thus the subsequent performance is approximated by

$$\forall \;\; \boldsymbol{p} \in \mathcal{P} \;\; y(\boldsymbol{p}) \in [y_{\min}, y_{\max}] \tag{2.10}$$

For a vector valued function $\boldsymbol{y}(\boldsymbol{p})$, the above procedure can be carried out for each element $y_i(\boldsymbol{p})$ of $\boldsymbol{y}(\boldsymbol{p})$.

Example By an optimization-based method we can find the trajectory that differs from the nominal trajectory on the interval $[0, T]$ by some measure. Fix the time t to T in (2.4) and the measure can be written as

$$y(\boldsymbol{p}) = \frac{1}{T}\int_0^T \parallel \boldsymbol{x}(\tau, \boldsymbol{p}) - \hat{\boldsymbol{x}}(\tau, \hat{\boldsymbol{p}}) \parallel \mathrm{d}\tau \tag{2.11}$$

Note that $y(\boldsymbol{p})$ is a nonnegative function and that the minimum is obtained at the nominal parameter value $\hat{\boldsymbol{p}}$.

$$\begin{aligned} y_{\min}(\boldsymbol{p}_{\min}) &= y(\hat{\boldsymbol{p}}) \\ &= \mathbf{0} \end{aligned} \tag{2.12}$$

It remains to find the maximum value of $y(\boldsymbol{p})$ and the corresponding parameter vector i.e. $y_{\max}(\boldsymbol{p}^*_{\max}) = \max_{\boldsymbol{p}\in\mathcal{P}} y(\boldsymbol{p})$ which can be done, at least approximately, by an appropriate optimization algorithm. Finally we have a measure of how much the trajectories on the time interval $[0, T]$ can differ from each other

$$y(\boldsymbol{p}) \in [0, y_{max}], \quad \boldsymbol{p} \in \mathcal{P} \tag{2.13}$$

using the measure (2.11).

3. Model

In order to evaluate the proposed SA methods a case study was defined. The chosen case for this study was a simplified model of a 3-dimensional (3-D) *Surface to Air Missile (SAM).*

3.1 Model

In this study a model of a SAM and an aerial target was used. The missile is guided by a proportional navigation control law. The system is described in more detail in [11]. The SAM is modeled as a discrete mass point system with geometrical and turning constraints. The trajectory of the aerial target is parameterised and was chosen to follow a horisontal line at two different altitudes, see Figure 3.1.

The model was made dynamic by introducing a third order filter which lags the system and mimic the longitudinal short period and the target seeker dynamic behavior of the missile.

The missile simulation model is generic in the sense that it is able to model and simulate different guidance laws and also that some missile specific data such as weight, impulse, burning time etc. can be adjusted to the specific missile modelled. The output from the simulation of the system is the state trajectories and some additional output signals such as miss distances, time to target etc.

The original model has been modified in this study. The main modifications are related to the aerodynamic of the missile, introducing a more complex aerodynamic model with respect to changes in the Mach number. Some modifications were also made in order to break the algebraic loops and improve the numerical stability and efficiency in the original model. The model has been implemented in three different ways, C-code, MATLAB [7] code and in SIMULINK [8].

3.2 Case study

In order to define the case, or benchmark, to be studied by different SA methods, a set of model parameters and output signals were selected. The model parameters $\boldsymbol{p}$ and their corresponding limits, the parameter set $\mathcal{P}$, were selected based on previous experience with missile systems. The set of selected model parameters and their limits can be found in Table 3.1. As output signals, $\boldsymbol{y}(t, \boldsymbol{p})$, acceleration ($a_{tot}$), missile total velocity (V_T), time to target (T_{final}) and missile trajectory (x_{final}) were selected. The simulated output signals that have been studied are presented in Table 3.2.

The total acceleration is defined as:

$$a_{tot} = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{3.1}$$

and the total velocity of the missile as:

$$V_T = \sqrt{V_x^2 + V_y^2 + V_z^2} \tag{3.2}$$

plus the T_{final} which is the time the missile needs to intercept the target and x_{final} which is the position of the missile when the fuse is activated by the target. Obviously

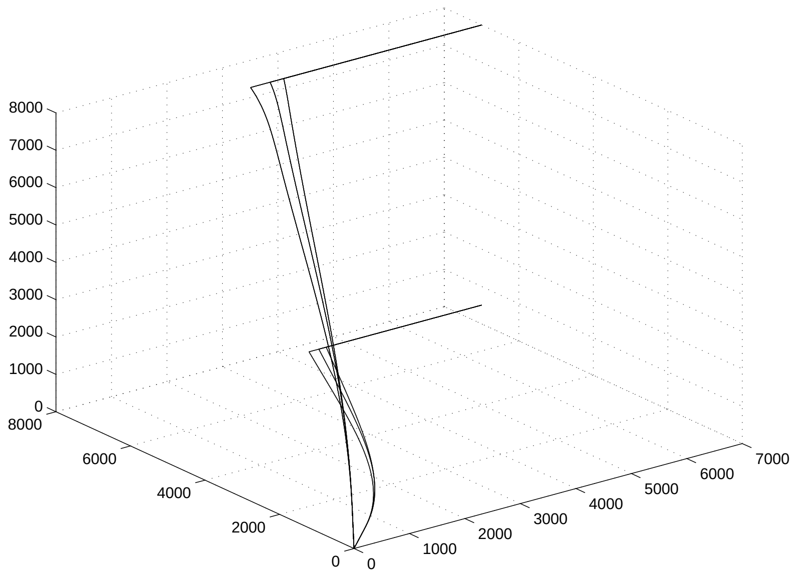

Figure 3.1: Trajectories of simulations of the missile to target system.

Parameter	Nom. Value	Limits	Unit	Description
$C_{D_{Unc}}$	1.0	±0.2	–	Drag coeff.
$C_{L_{Unc}}$	1.0	±0.2	–	Lift coeff.
$freq$	8.0	±2.0	rad/s	Closed loop frequency
$damp$	0.7	±0.2	–	Closed loop damping
$SpecImp_1$	2000.0	±300	Ns/kg	Specific impulse for eng. no. 1
T_{Burn_1}	1.0	±0.2	s	Burning time for eng. no. 1
$SpecImp_2$	2000.0	±300	Ns/kg	Specific impulse for eng. no. 2
T_{Burn_2}	10.0	±2.0	s	Burning time for eng. no. 2

Table 3.1: Studied model uncertainty parameters, $\boldsymbol{p}$

Result	Unit	Description
a_{tot}	m/s^2	Total acceleration
V_T	m/s	Total velocity
T_{final}	s	Time to target
x_{final}	m	Trajectory

Table 3.2: Studied output signals, $\boldsymbol{y}(t, \boldsymbol{p})$

the first two output signals are available during the entire simulation, and the latter two are derived from the final state of the system.

The analysed model, SAM, can be described as a set of time dependent first order nonlinear differential equations with parametric uncertainties. The original model was a set of differential-algebraic equations that included switches that made the model discontinuous. The algebraic loops in the model, were eliminated by adding additional fast first order lag filters.

3.3 Numerical methods

A model based on *ordinary differential equations (ODEs)* is often simulated by using numerical integration methods. As a part of a verification process of the implementation of the model, it is natural to verify the correctness of the computations.

Our intention here is to point out numerical problems that arose in the study of the model discussed in the case study, but is of general interest. The theory part is introduced in order to give support for the discussed issues. For introduction to computer methods for ordinary differential equations and differential algebraic equations, see for example [2].

Numerical solution of ordinary differential equations We will study numerical solution of ordinary differential equations

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \quad a < t < b \tag{3.3}$$

with initial values

$$\boldsymbol{x}_0 = \boldsymbol{x}(t_0) = \boldsymbol{c} \tag{3.4}$$

In order to make an approximation we discretize the interval at N discrete points

$$a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b \tag{3.5}$$

and seek approximations of $\boldsymbol{x}$ at those points

$$\boldsymbol{x}_i \approx \boldsymbol{x}(t_i), \quad i = 0, \ldots, N \tag{3.6}$$

note that equality holds for $i = 0$. We define the step size as

$$h_n = t_n - t_{n-1} \tag{3.7}$$

One simple numerical integration method is Euler's method, which we now will derive. Expand $\boldsymbol{x}(t_n)$ in a Taylor series around t_{n-1}

$$\boldsymbol{x}(t_{n-1} + h_n) = \boldsymbol{x}(t_{n-1}) + h_n \dot{\boldsymbol{x}}(t_{n-1}) + O(h_n^2) \tag{3.8}$$

We define forward Euler's method by substituting the true values $\boldsymbol{x}(t_{n-1} + h)$ and $\boldsymbol{x}(t_{n-1})$ by the approximations $\boldsymbol{x}_n$ and $\boldsymbol{x}_{n-1}$ and ignoring higher order terms

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + h_n \boldsymbol{f}(t_{n-1}, \boldsymbol{x}_{n-1}) \tag{3.9}$$

Euler's method is often used to illustrate basic concepts, but for computations more advanced methods are often used.

Piecewise continuous function $\boldsymbol{f}(t, \boldsymbol{x})$ Numerical integration methods are usually developed with the assumption that $\boldsymbol{f}(t, \boldsymbol{x})$ and its derivatives are continuous on the interval $[t_{n-1}, t_n]$ up to a certain order. Piecewise continuous functions can be dealt with by selecting discretisation points at every point of discontinuity, and thus avoid the problem.

For example, the mass as a function of time in the model of the case study has a piecewise continuous derivative with discontinuities at the points $t = 0\text{s}, 1\text{s}, 11\text{s}$, and these points should be among the discretisation points in order to avoid problem caused by the mass function.

Memory block In the SIMULINK model we have an "algebraic loop" of the algebraic variable α. The dependencies are in the following form

$$\begin{aligned} &\vdots \\ z &= g_1(\boldsymbol{x}, \alpha) \\ \alpha &= g_2(\boldsymbol{x}, z) \\ &\vdots \end{aligned} \tag{3.10}$$

where $\boldsymbol{x}$ is a differential variable, z and α are internal algebraic variables

One way to resolve these dependencies of α in the simulink model is to put in a memory block. The cause of using a memory block for α is that we keep two copies of α in the numerical solution of the system, one in the current time step α_{n-1} and one in the previous time step α_{n-2}

$$\begin{aligned} &\vdots \\ z_{n-1} &= g_1(\boldsymbol{x}_{n-1}, \alpha_{n-2}) \\ \alpha_{n-1} &= g_2(\boldsymbol{x}_{n-1}, z_{n-1}) \\ &\vdots \end{aligned} \tag{3.11}$$

If α varies slowly with time this strategy might work well. One problem however is that the model is mixed with the numerical integration method, and it is hard to know what is what. The subsequent analysis of numerical methods and modelling issues becomes more difficult.

However, in this case it's a rather easy task to rewrite the problem into one of the following form

$$0 = g(\boldsymbol{x}, \alpha) \tag{3.12}$$

To use this model in a numerical integration algorithm is straight forward. Solve for α_n in every time step. For Euler's method, the overall algorithm becomes

$$0 = g(\boldsymbol{x}_{n-1}, \alpha_{n-1}), \quad \text{solve for } \ \alpha_{n-1} \tag{3.13}$$

$$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + h\boldsymbol{f}(t, \boldsymbol{x}_{n-1}, \alpha_{n-1}) \tag{3.14}$$

4. Simulation

4.1 Local methods results

In order to investigate the sensitivity to variations in parameters of the SAM simulation model, a local sensitivity analysis was conducted.

The model can be formulated mathematically in (2.2) with the parameter vector defined as:

$$\boldsymbol{p} = \left[\begin{array}{cccccccc} C_{D_{Unc}} & C_{L_{Unc}} & freq & damp & SpecImp_1 & T_{Burn_1} & SpecImp_2 & T_{Burn_2} \end{array}\right] \tag{4.1}$$

and the output signal $\boldsymbol{y}(t, \boldsymbol{p})$, from the system, defined as:

$$\boldsymbol{y}(t, \boldsymbol{p}) = \left[\begin{array}{cccc} a_{tot} & V_T & T_{final} & x_{final} \end{array}\right] \tag{4.2}$$

Obvious the first two output signals are available during the entire simulation, and the latter two are derived from the final state of the system. The entries in the sensitivity matrix $\boldsymbol{T}$ were approximated by (2.6) which are normed compared to (2.5) in order to get the *relative* sensitivity of the parameters. The calculations where conducted by simulating the output signals from the system (2.2) with each of the parameters disturbed individually by a small amount which resulted in $n + 1$ calls to the model, where n is the number of uncertainty parameters, in this case 8. The sensitivity was then approximately calculated using (2.6) in each time step along the state trajectory in order to get the result for a_{tot} and V_T. The findings can be seen in Figure 4.1 to Figure 4.4. The analysis for T_{final} and x_{final} was conducted using the final state of the system. The results of the local sensitivity analysis is summarised in (4.3) and (4.4) where the $\max(\boldsymbol{T}(t))$ and $\min(\boldsymbol{T}(t))$ are shown.

$$\boldsymbol{T}_{\max} = \left[\begin{array}{llllllll} 0.044 & 0.372 & 0.298 & 1.874 & 0.555 & 0.245 & 0.179 & 0.788 \\ 0 & 0.021 & 0.002 & 0.000 & 0.998 & 0 & 0.380 & 0 \\ 0.019 & 0 & 0 & 0 & -0.312 & 0.312 & -0.188 & 0.188 \\ -0.014 & 0.014 & 0.000 & 0.000 & 0.182 & -0.205 & 0.074 & -0.074 \end{array}\right] \tag{4.3}$$

$$\boldsymbol{T}_{\min} = \left[\begin{array}{llllllll} -0.041 & 0 & -0.411 & -0.461 & -0.192 & -0.556 & -0.784 & -0.048 \\ -0.039 & 0.000 & 0.000 & -0.003 & 0 & -0.988 & 0 & -0.377 \\ 0.019 & 0 & 0 & 0 & -0.312 & 0.312 & -0.187 & 0.187 \\ -0.014 & 0.014 & 0.001 & 0.001 & 0.182 & -0.205 & 0.074 & -0.074 \end{array}\right] \tag{4.4}$$

A closer examination of the results presented in (4.3) and (4.4) and in Figure 4.1 to Figure 4.4 reveals:

1. The model discontinuity when switching from boost- to sustainphase at 1s is clearly visible in Figure 4.1 to Figure 4.4.

2. The parameters $freq$ and $damp$ have only a minor impact on the total velocity of the missile.

3. The parameters $C_{L_{Unc}}$, $freq$ and $damp$ have no impact on T_{final}.

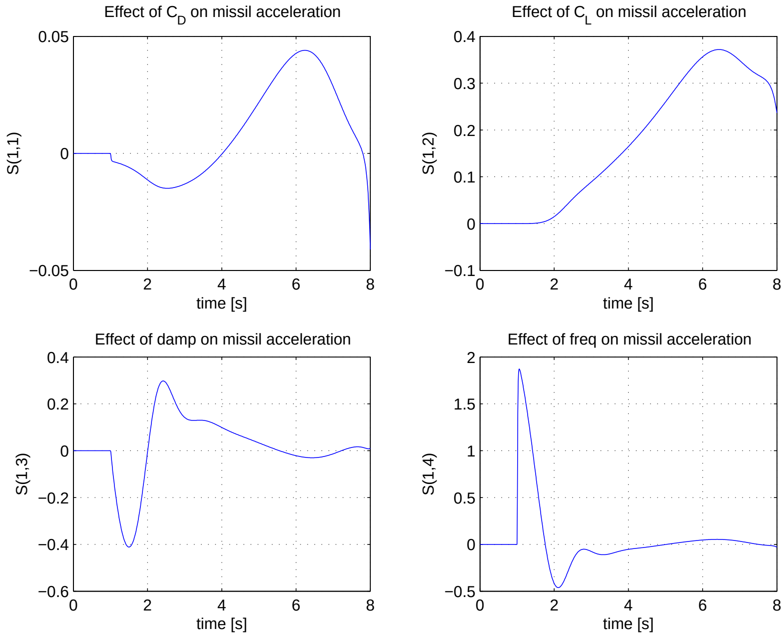

Figure 4.1: Time histories for some elements of the sensitivity matrix, results for a_{tot}, elements T[1,1] to T[1,4].

4. The relative sensitivity of $SpecImp_1$ has almost twice the impact on T_{final} compared to $SpecImp_2$.

4.2 Global methods results

In order to assess the global sensitivity of the case study a sample-based SA was conducted. The sample-based SA was made for two different cases, described in chapter 3, where the samples are uniformly distributed in the admissible parameter set $\mathcal{P}$. All the input parameters were randomly sampled in 10000 simulations. Simulations have also been done, where one input parameter has been randomly sampled. All combinations of the input parameter extreme values, have finally been simulated. The simulations have been made at two altitudes, 500m and 8000m. The target has a horisontal trajectory at constant speed, 300m/s. For other model input data, default values have been used. A selected number of simulated cases are presented in Figure 4.5 to Figure 4.15.

In Figure 4.5 the input parameter T_{Burn_2} is plotted versus the T_{final}. In this case, the input parameter T_{Burn_2} is randomly sampled and all the others are kept constant at their nominal values. The input parameter T_{Burn_2} is varied between 8s and 12s and the resulting time to target (T_{final}) is between 9.4s to 10.1s. It can clearly be noted that a higher burn time of powder will result in a lower acceleration, which leads to a higher T_{final}. It can also be seen that there is approximately a linear dependence between T_{Burn_2} and the T_{final}. In Figure 4.6, the result from the simulations, where all input parameters are randomly sampled, is presented. The input parameter T_{Burn_2} is projected and drawn versus the T_{final}. In this data, an upward trend tendency can be observed as in Figure 4.5. The distribution of approximately 2s in T_{Burn_2} is caused by the total variation of all parameters in the parameter set $\mathcal{P}$. In the results presented in Figure 4.7, the input parameters are sampled by taking all combinations of the extreme input parameters and nominal values within $\mathcal{P}$, for the case of 500m altitude. The same distribution of approximately 3s, as can be observed in Figure 4.6, are presented here.

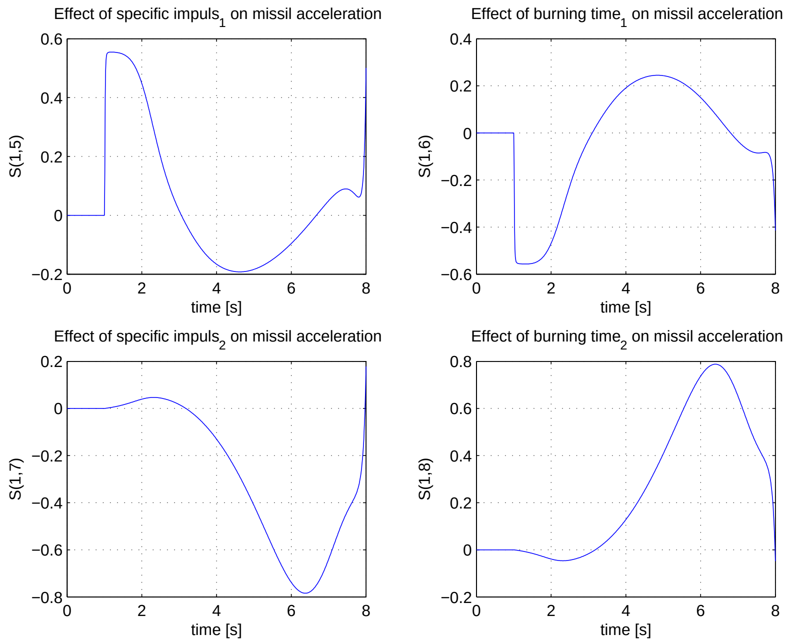

Figure 4.2: Time histories for some elements of the sensitivity matrix, results for a_{tot}, elements T[1,5] to T[1,8].

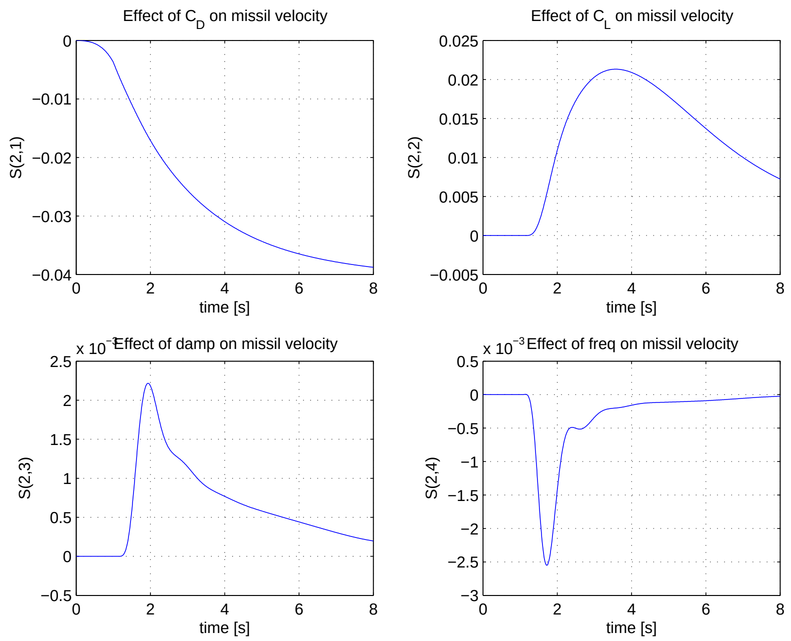

Figure 4.3: Time histories for some elements of the sensitivity matrix, results for V_T, elements T[2,1] to T[2,4].

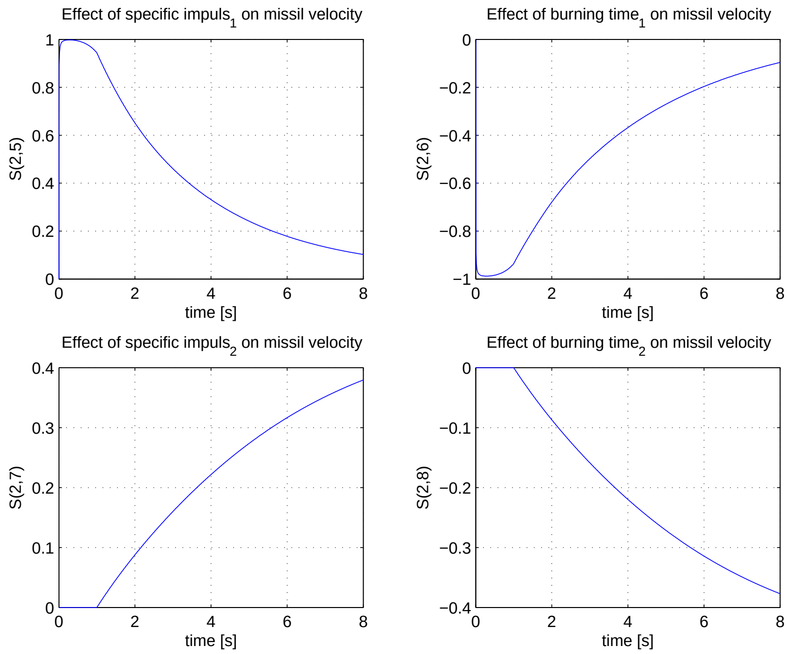

Figure 4.4: Time histories for some elements of the sensitivity matrix, results for V_T, elements T[2,5] to T[2,8].

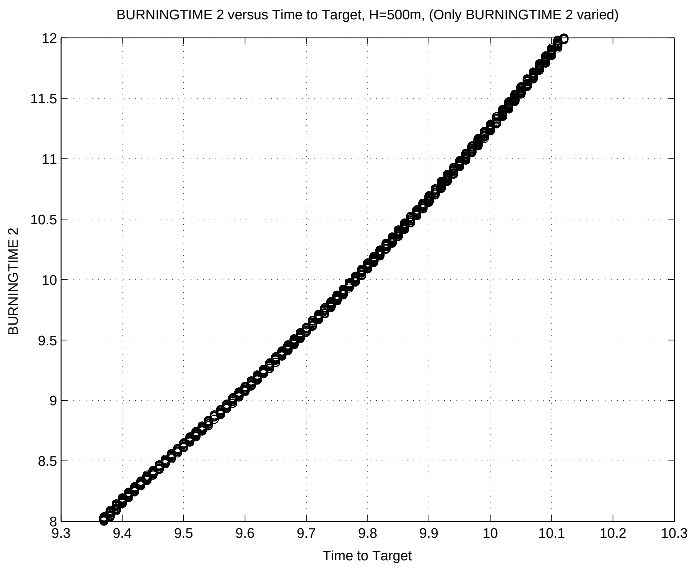

Figure 4.5: T_{Burn_2} versus time to target, T_{final} where only T_{Burn_2} is randomly sampled. $H = 500$m.

In Figure 4.8 the input parameter $C_{D_{Unc}}$ is plotted versus the T_{final}. In this simulation, the input parameter $C_{D_{Unc}}$ is randomly sampled from a rectangular distribution while the others are kept at their nominal values. $C_{D_{Unc}}$ is allowed to be between 0.8 and 1.2 and the resulting T_{final} lies between 9.3s to 10.4s. As in the previous case, see Figure 4.6, there is approximately a linear dependence between $C_{D_{Unc}}$ and the T_{final}. As shown in Figure 4.8, an increase in the drag of the missile will result in a longer flight time, i.e. higher T_{final}. In the simulation presented in

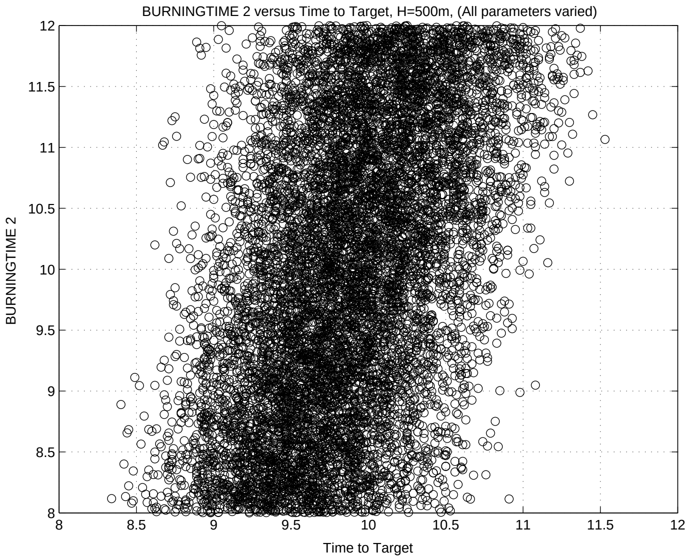

Figure 4.6: T_{Burn_2} versus time to target, T_{final} where all input parameters are randomly sampled. $H = 500$m.

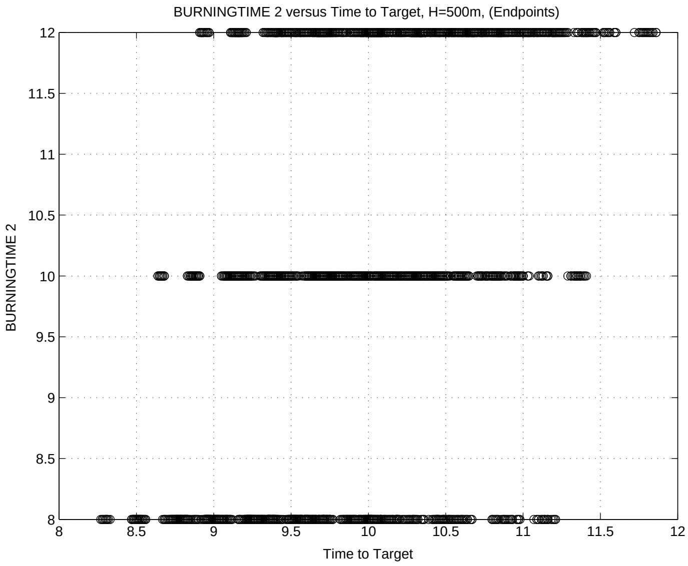

Figure 4.7: T_{Burn_2} versus time to target, T_{final} where the input parameter endpoints are randomly sampled. $H = 500$m.

Figure 4.9 , all input parameters are randomly sampled over the admissible parameter set, $\mathcal{P}$. The input parameter $C_{D_{Unc}}$ is projected and plotted versus the T_{final}. In the data presented, an upward trend tendency can be noted. The parameter $C_{D_{Unc}}$ is sampled between 0.8 and 1.2 and the resulting T_{final} is distributed approximately 2s. In the simulation presented in Figure 4.10 , the input parameters are sampled by taking all combinations of the extreme input parameters, i.e. the corner points in $\mathcal{P}$, and normal values. The input parameter $C_{D_{Unc}}$ is plotted versus the T_{final}. The input parameter $C_{D_{Unc}}$ is sampled between 0.8 and 1.2 and the T_{final} is distributed approximately 2.5s.

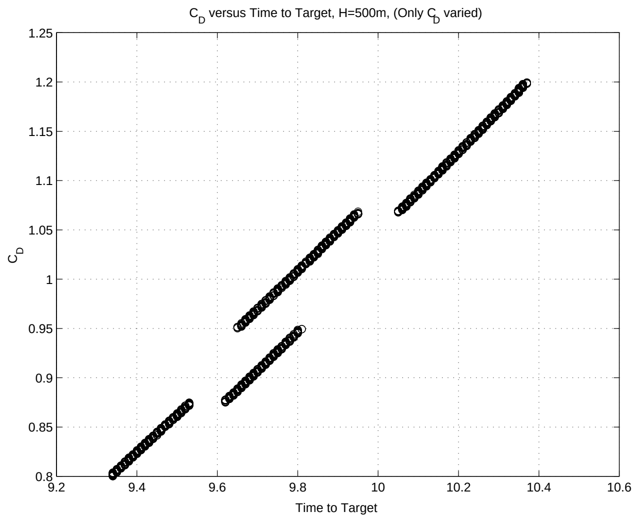

Figure 4.8: $C_{D_{Unc}}$ versus time to target, T_{final} where only $C_{D_{Unc}}$ is randomly sampled. $H = 500$m.

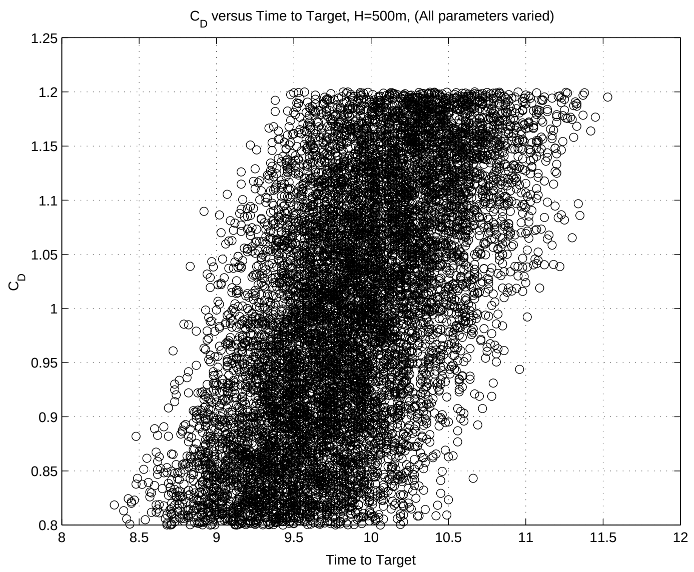

Figure 4.9: $C_{D_{Unc}}$ versus time to target, T_{final} where all input parameters are randomly sampled. $H = 500$m.

In Figure 4.11 and 4.12 , the effect of the closed loop damping *damp* and frequency $freq$ of the missile on the T_{final} is shown. The input parameter *damp* is sampled between 0.5 and 0.9 and the $freq$ between 6rad/s and 10rad/s. The distribution in the data caused by the variation of the other parameters are approximately 3s in both cases. The overall conclusion drawn out of Figure 4.11 and Figure 4.12 , is that the sensitivity of T_{final} with respect to *damp* and $freq$ is negligible. An interesting observation can be made by comparing this result with the result from the local analysis, see 4.3 and 4.4 , which entries are zero.

In Figure 4.13 the input parameter $SpecImp_2$ is plotted versus the T_{final}. In this simulation, the input parameter $SpecImp_2$ is randomly sampled and the others are

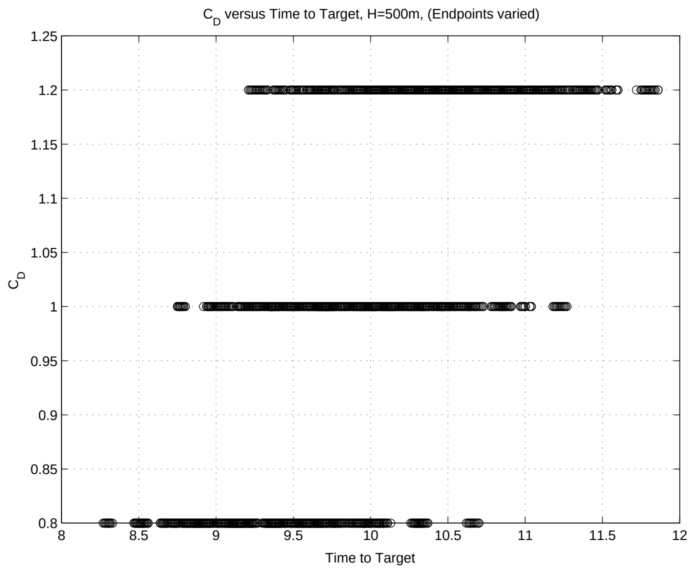

Figure 4.10: $C_{D_{Unc}}$ versus time to target, T_{final} where the input parameter endpoints are randomly sampled. $H = 500$m.

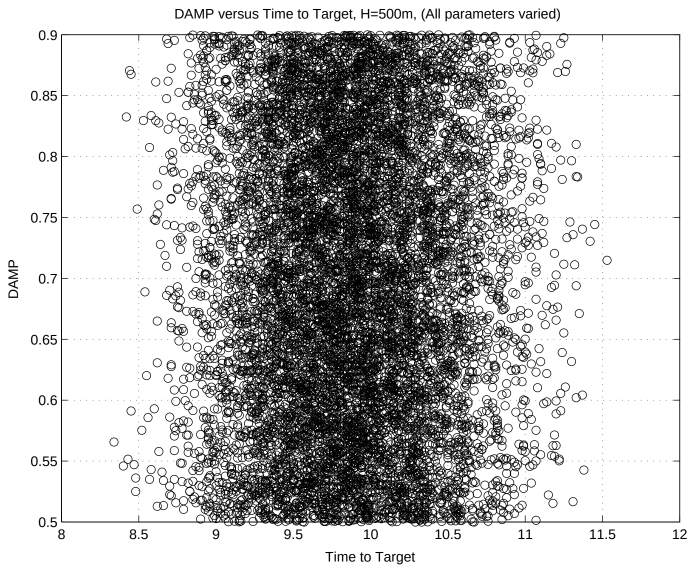

Figure 4.11: *damp* versus time to target, T_{final} where all input parameters are randomly sampled. $H = 500$m.

kept constant at their nominal values. The input parameter $SpecImp_2$ is sampled between 1700Ns/kg and 2300Ns/kg and the T_{final} is simulated between 9.5s to 10.1s. Also in this case there is approximately a linear dependence between $SpecImp_2$ and the T_{final}. In the simulation presented in Figure 4.14, all input parameters are randomly sampled. The input parameter $SpecImp_2$ is projected and plotted versus the T_{final}. In the data shown, a downward trend tendency, with a distribution of approximately 2s, can be noted. The conclusion drawn from the SA is that a higher $SpecImp_2$, which means a higher thrust, will result in a shorter time to target, T_{final}, a result that is physically reasonable.

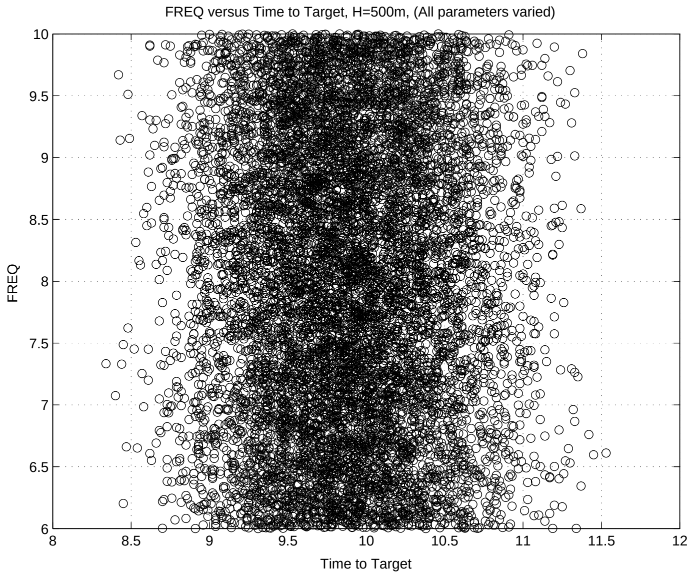

Figure 4.12: $freq$ versus time to target, T_{final}, where all input parameters are randomly sampled. $H = 500$m.

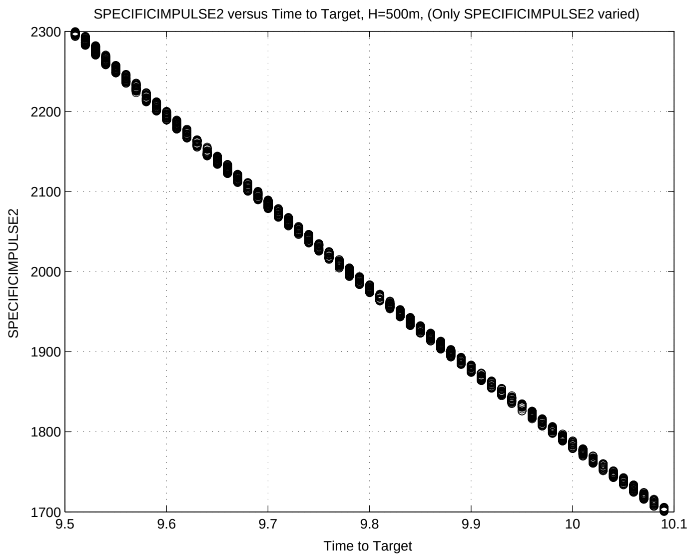

Figure 4.13: $SpecImp_2$ versus time to target, T_{final} where only $SpecImp_2$ is randomly sampled. $H = 500$m.

In the simulation presented in Figure 4.15, all input parameters are randomly sampled. The input parameter $SpecImp_1$ is projected and plotted versus the norm of the total velocity, V_T, in one point. In this data, an upward trend tendency can be noted. The input parameter $SpecImp_1$ is sampled between 1700Ns/kg and 2300Ns/kg and V_T is distributed approximately 150m/s.

In the simulations the norm of the total accelerations are calculated, which lead to strange results. This is believed to depend on the fact that the lateral and longitudinal accelerations in the model are not separated.

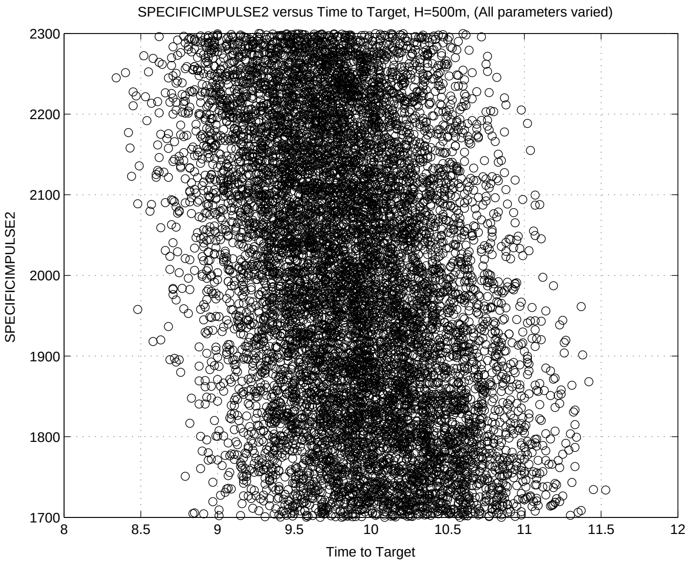

Figure 4.14: $SpecImp_2$ versus time to target, T_{final} where all input parameters are randomly sampled. $H = 500$m.

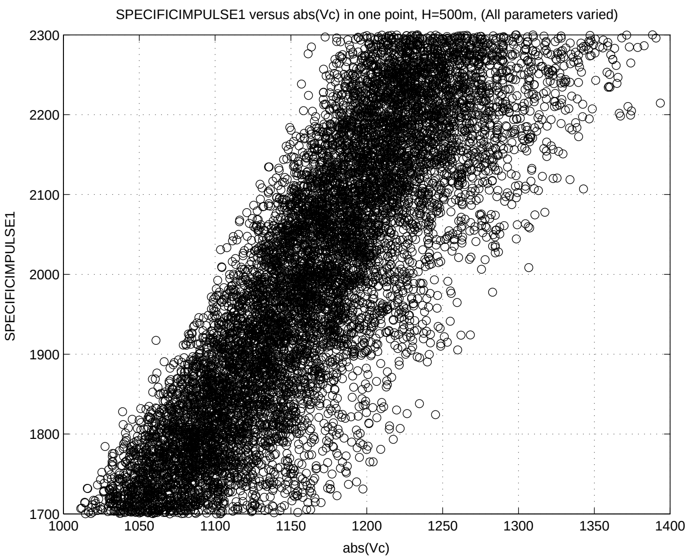

Figure 4.15: $SpecImp_1$ versus total velocity, V_T where all input parameters are randomly sampled. $H = 5000$m.

5. Discussion

5.1 Conclusion

This study was conducted within the project Technical Threat Systems Analysis. The main purpose of this study has been to make a preliminary investigation of the possiblity of using SA methods for analysing simulation models. The work presented in this report clearly indicates that the applied methods for SA have some potential features that will be beneficial for future studies within this project. Some items are worth highligting:

- Some conclusion regarding the general applicablity of local versus global SA methods can not be made based upon the results obtained from the analysis of the SAM system, due to the low complexibility in the model. In the global analysis made on the case study, the correlation between the input and output signals were more or less linear, which indicates that a local analysis would result in similar results as the global analysis.
- Local SA methods used along the state trajectory can be used to analysing time dependent non-linear systems. Discontinuities in the model, caused by the switches, did not caused any major obstacles for the analysis.
- In order to trust the results from SA, it is important that the model is verified, for example with respect to numerical computations.
- The overall conclusion is that SA has a strong potential to be a useful tool in evaluating a model in terms of how parametric uncertainties affects output of missile systems.

Due to the quality of the generic model used in the case study, no general conclusions can be made on how the chosen parameters affect the performance of other missile to target systems. Since this study was mainly focused on an initial evaluation of some methods for SA using a simplified model of a SAM, the future research should be oriented toward the application of the analysis methods on more realistic, and complex problems using more complete models of aircraft-missile systems as case studies. It should also be focused on developing and the application on new promising methods, based on, e.g. optimization and more efficient numerical algorithms for solving complex SA problems.

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