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A Generic Model of Aircraft Dynamics

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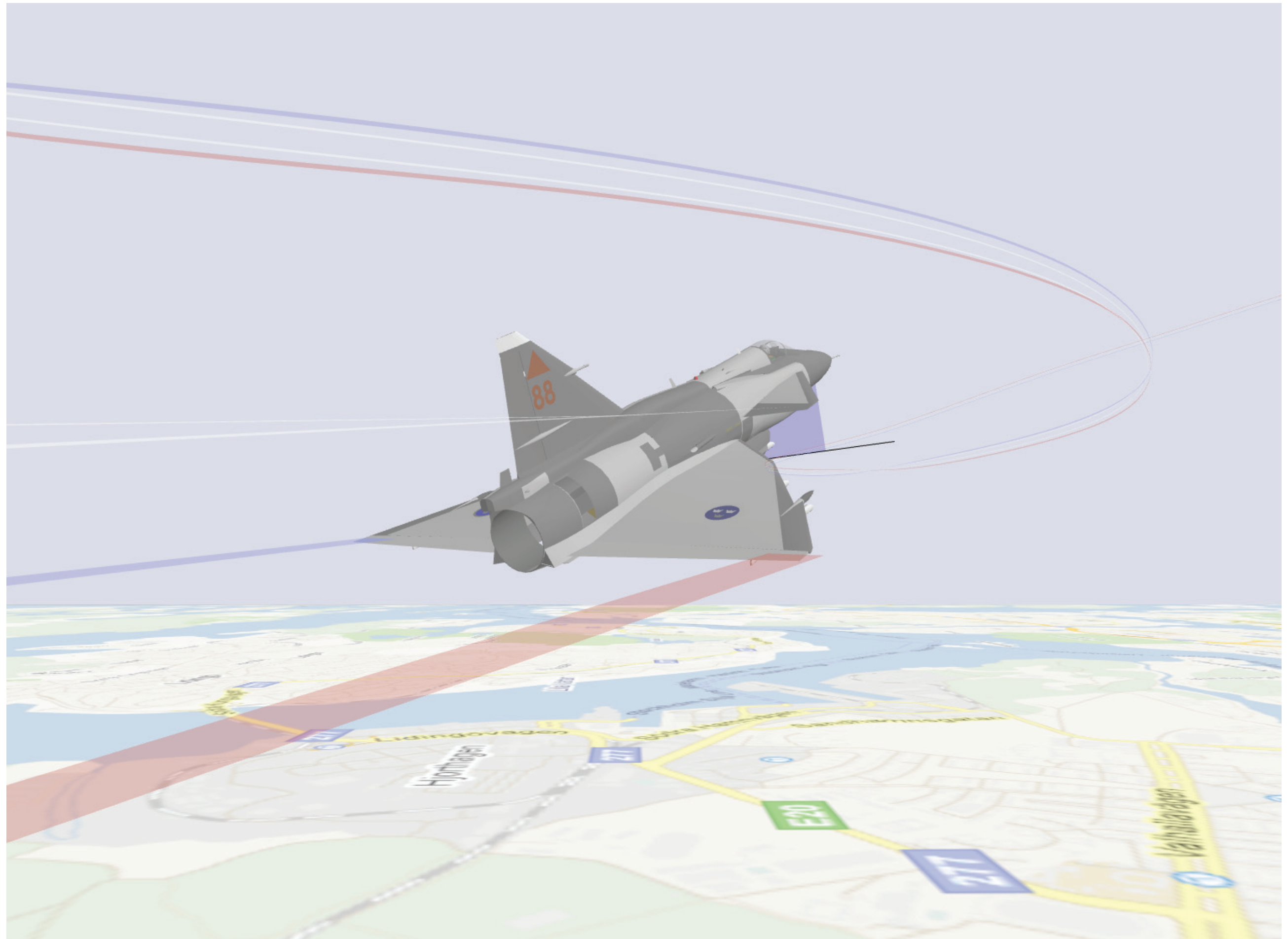

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Abstract

This report describes a generic aircraft model which is based on a simplification of the full rigid body equations of motion most often used for realistic simulations. The model is applicable to piloted simulations as well as modeling of autonomous behavior of aircraft when there is a path planner och behavior generator present. For the latter types of applications a simple velocity vector following autopilot is included in the model . Particular emphasis has been put on making visible the various assumptions used when obtaining the model. This makes it easy to adapt the model to other applications, such as bank-to-turn operated missiles and aircraft with nonstandard configurations.

Keywords

Flight mechanics, Flight dynamics, Aircraft model

Sammanfattning

Rapporten beskriver en generisk flygplansmodell som baseras på en förenkling av den fulla stelkroppsdynamiken som oftast används vid realistiska simuleringar. Modellen är tillämpbar både för pilotstyrda simuleringar och modellering av autonomt beteende i de fall då det finns en planeringsfunktion eller beteendegenerator tillgänglig. För de senare typerna av tillämpningar finns det inkluderat i modellen en enkel autopilot som klarar att följa commandon för hastighetsvektorn. Speciell vikt har lagts vid att synliggöra de olika antaganden som använts vid framtagandet av modellen. Detta gör det lätt att anpassa modellen till andra tillämpningar, såsom "bant-to-turn"-opererade missiler och flygplan med ickestandard konfiguration.

Nyckelord

Flygmekanik, Flygplansmodell

Contents

1 Introduction

In this report we develop a generic mathematical model of an aircraft flying in coordinated flight.[1] The model is indented to serve as a platform for simulations as well as performance assessments and can be used in (human) piloted simulations or in an autopilot guided mode. It is is based on linearized equations of motion in all three axes (a 6 degree-of-freedom model) and can be made to represent a large number of aircraft by changing the (relatively) few parameters in it. An underlying assumption of this is that the aircraft to be modeled is either "conventional" in its (open loop, bare airframe) dynamics, or, in case of an aircraft with a stability augmenting or dynamics synthesizing control system, has been rendered "conventional" in its closed loop behavior (i.e. with flight control system engaged).

1.1 Outline

In the next chapter we present a summary of the model which can be used as a reference and lookup when implementing it and in the following chapters we present a detailed derivation of it.

1.2 Notation

We use standard mathematical notation where (real or complex) scalars are marked with ordinary typeface, like m, and vectors in $\mathbb{R}^n$ and matrices in $\mathbb{R}^{n\times m}$ are indicated by bold typeface, like $\boldsymbol{x}$ and $\boldsymbol{A}$. Unless otherwise indicated, vectors are considered as column vectors and the elements of vectors and matrices are indicated by subscripts. Transposition of a vector or matrix is indicated by a superscript T and the norm (always the 2-norm) of a vector is marked as $\|\cdot\|$. When the elements of a vector are explicitly listed together they are enclosed by square brackets like $\boldsymbol{x} = [x_1, x_2]^T$. A square diagonal matrix $\boldsymbol{D} \in \mathbb{R}^{n\times n}$ with the diagonal elements $d_1, \ldots, d_n$ is indicated as $\boldsymbol{D} = \mathrm{diag}(d_1, \ldots, d_n)$.

Superscripts within parentheses are often used to enumerate individual elements of a family of vectors or scalars, like $\boldsymbol{f}^{(a)}$. The linear subspace of $\mathbb{R}^n$ spanned by m vectors $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(m)} \in \mathbb{R}^m$ is denoted by $[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(m)}]$ and the orthogonal complement, i.e. the set $\{\boldsymbol{y} \in \mathbb{R}^n \mid \boldsymbol{y}^T\boldsymbol{x}^{(j)} = \boldsymbol{0},\ j = 1, \ldots, m\}$, is denoted $[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(m)}]^\perp$. Unit length vectors in $\mathbb{R}^3$ pointing along the (positive direction of) coordinate axes in coordinate systems will occur below and they are denoted as $\boldsymbol{e}_j$ where the subscript $j \in \{1, 2, 3\}$ refer to the coordinate axis in question. More generally, for any vector $\boldsymbol{x} \in \mathbb{R}^n \setminus \boldsymbol{0}$ we define $\boldsymbol{e}_{\boldsymbol{x}} = (1/\|\boldsymbol{x}\|)\boldsymbol{x}$. The cross product between vectors in $\mathbb{R}^3$ is marked with $\times$.

The imaginary unit is marked i. Time differentiation of a (time differentiable) vector $\boldsymbol{x}$ is marked with the dot notation as $\dot{\boldsymbol{x}}$. Often the dependence in a certain function on some variables is suppressed in the notation to avoid cluttering the presentation, but when this is the case it should be clear from the context. In particular, the time argument t implicit in many quantities is almost never written out.

[1] The model is also applicable to missiles which are operated in bank-to-turn mode. In (Robinson, 2010) a similar model is developed which is applicable for skid-to-turn operated missiles. The material presented here builds on material in (Robinson, 2010).

1.3 Acknowledgment

The contents of the report has been much influenced by Mr. Peter Strömbäck and Mr. Emil Salling, both at FOI. Mr. Salling implemented and tested the model in the framework `Merlin` (Salling *et al.*, 2009) with the aid of Mr. Strömbäck. The latter also did reference implementations of parts of the model (including the autopilot) in `Matlab` which resulted in the correction of several errors in the manuscript and improvements of the formulations used. For this the author is very grateful. The author is also indebted to Dr. Petter Ögren at FOI for help with proofreading of the manuscript. Dr. Ögren suggested several changes to the manuscript which increased clarity and readability.

2 Summary of model

In this chapter we give a summary of the equations and relations needed to implement the model and in the following chapters we provide a detailed derivation of these. The contents of the various subsections of Section 2.1 can serve as a reference and the main assumptions underlying the derivation of the model are listed in Section 2.1.7. In Section 2.2 a summary of a simple velocity vector following autopilot is given.

2.1 Aircraft dynamics

We assume that the aircraft behaves as rigid body and that with it is associated a body fixed Cartesian coordinate system B with the (flight mechanical) standard orientation (Stevens & Lewis, 2003); the x-axis is directed along the main axis of the aircraft with positive extension forward, the y-axis points out over the right wing and the z-axis points downward. We moreover assume that the aircraft is left-right symmetric and that it exhibits coordinated flight (Phillips, 2010), so that the apparent force in the y-direction in B (the sideforce) felt by a pilot is zero at all times. This is the same as to say that the aerodynamic force in the y-direction in B is zero and by the symmetry of the airframe the sideslip angle must be zero at all times.

The relations listed here describe the part of the model which corresponds to a piloted aircraft, with variables expressed[1] in the frame B. In order to get a full model representing flight in an Earth fixed frame E, where position and orientation should be defined, a relation describing the evolution of the orientation must be added and the relation for velocity in E must be integrated.

2.1.1 Definitions

The velocity vector $\boldsymbol{v}$ and angular velocity vector $\boldsymbol{\omega}$ in B have components given by $\boldsymbol{v} = [u, v, w]^T$ and $\boldsymbol{\omega} = [p, q, r]^T$. From these, the angle of attack α, the airspeed V and the wind axis roll rate $p^{(W)}$ are defined as

$$\begin{aligned}
\alpha &= \arctan(w/u), & (2.1)\\
V &= \sqrt{u^2 + v^2 + w^2}, & (2.2)\\
p^{(W)} &= \frac{1}{V}\boldsymbol{v}^T\boldsymbol{\omega}. & (2.3)
\end{aligned}$$

The force vector in B is denoted $\boldsymbol{f} = [f_x, f_y, f_z]^T$ and we divide it into aerodynamic, gravity and thrust induced components, respectively, as

$$\begin{aligned}
f_x &= f_x^{(a)} + f_x^{(g)} + f_x^{(t)},\\
f_y &= f_y^{(a)} + f_y^{(g)} + f_y^{(t)},\\
f_z &= f_z^{(a)} + f_z^{(g)} + f_z^{(t)},
\end{aligned}$$

where, by assumptions (given in this chapter), $f_y^{(a)} = f_y^{(t)} = f_z^{(t)} = 0$. From these components we define the lift and drag forces f_L, f_D, respectively, as

$$\begin{aligned}
f_L &= f_x^{(a)}\sin(\alpha) - f_z^{(a)}\cos(\alpha), & (2.4)\\
f_D &= -f_x^{(a)}\cos(\alpha) - f_z^{(a)}\sin(\alpha). & (2.5)
\end{aligned}$$

[1] We don't give units for the variables, it is assumed that this is done at time of implementation in a consistent system of units, e.g. the metric (SI) system.

2.1.1.1 Basic forces

Lift

The lift force f_L can be expressed in terms of the aerodynamic lift force coefficient C_L as

$$f_L = \frac{1}{2}\rho V^2 S_{ref} C_L(\alpha), \tag{2.6}$$

where ρ is the air density and S_{ref} is the reference (wing) area. We shall assume that f_L is linear in α which is equivalent to assuming linearity in α of C_L, viz

$$C_L(\alpha) = \frac{\partial C_L}{\partial \alpha}|_{\alpha=0}\,\alpha. \tag{2.7}$$

With this assumption we can write (2.6) as

$$f_L = \frac{1}{2}\rho V^2 S_{ref}\frac{\partial C_L}{\partial \alpha}|_{\alpha=0}\,\alpha. \tag{2.8}$$

Drag

The drag force f_D is modeled in terms of the drag force coefficient C_D as

$$f_D = \frac{1}{2}\rho V^2 S_{ref} C_D(\alpha, M), \tag{2.9}$$

where

$$\begin{aligned} C_D(\alpha, M) &= C_{D_0}(M) + K(M)C_L(\alpha)^2 \\ &= C_{D_0}(M) + K(M)\Big(\frac{\partial C_L}{\partial \alpha}|_{\alpha=0}\Big)^2\alpha^2 \end{aligned} \tag{2.10}$$

and the parasitic drag coefficient C_{D_0} and the induced drag quadratic factor K both have some prescribed dependencies on Mach number M. The value of α is obtained from the load factor η in (2.12) below (via $\eta^{(a)}$ in (2.12), using (2.15)).

Thrust

We assume that the thrust acts only in the (positive) x-direction in B so that $f_y^{(t)} = f_z^{(t)} = 0$. The thrust component $f_x^{(t)}$ in the x-direction is assumed to have the dynamics

$$\dot{f}_x^{(t)} = \frac{1}{\tau_T}(t_{ss} f_{T_0}(M,h) - f_x^{(t)}), \tag{2.11}$$

where $t_{ss} \in [0,1]$ is the throttle setting, $f_{T_0}(M,h)$ is the maximum engine thrust at Mach number M and altitude h, and τ_T is the engine response time constant.

2.1.1.2 Load factor

The aerodynamic load factor $\eta^{(a)}$ and (total) load factor η, respectively, are defined as

$$\eta^{(a)} = \frac{f_L}{mg}, \quad \eta = \eta^{(a)} - \frac{F_\alpha^{(gt)}}{g}, \tag{2.12}$$

where $F_\alpha^{(gt)}$ is a normalized force (acceleration) quantity given by

$$F_\alpha^{(gt)} = \frac{1}{m}\big(f_z^{(g)}\cos(\alpha) - (f_x^{(g)} + f_x^{(t)})\sin(\alpha)\big). \tag{2.13}$$

2.1.1.3 State variables

The dynamics in B of the aircraft is represented in terms of the six state variables $\eta, \dot{\eta}, V, p^{(W)}, q, r$, as described below.

2.1.2 Pitch channel

The dynamics in the pitch channel are modeled in terms of the load factor η, its derivative $\dot{\eta}$ and commanded load factor η_c as

$$\frac{d}{dt}\begin{bmatrix}\eta\\ \dot{\eta}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\omega_{sp}^2 & -2\zeta_{sp}\omega_{sp}\end{bmatrix}\begin{bmatrix}\eta\\ \dot{\eta}\end{bmatrix} + \begin{bmatrix}0\\ \omega_{sp}^2\end{bmatrix}\eta_c, \tag{2.14}$$

where the undamped natural frequency ω_{sp} and damping ζ_{sp} are either open loop ("bare airframe") values or closed loop values (synthesized e.g. by a stability augmentation system). In the former case the variable η_c must be considered as a scaled version of the actual stick input and in the latter case η_c can be considered as a rough approximation of a real stick input (without any filtering) or as the reference signal sent to the controller subsystem by an autopilot. In case of a closed loop interpretation, the values of ω_{sp}, ζ_{sp} are given by handling criteria. The values for α and q are computed from $\eta, \dot{\eta}$ using two algebraic relations, see below.

2.1.2.1 Angle of attack

From (2.7), (2.8) and (2.12) we have

$$\eta^{(a)} = \frac{1}{2mg}\rho V^2 S_{ref}\frac{\partial C_L}{\partial \alpha}|_{\alpha=0}\,\alpha \tag{2.15}$$

and from this α can be solved. However, the dynamics in (2.14) are formulated in terms of η and therefore α must be (nonlinearly) solved[2] from the rightmost relation in (2.12), given η.

2.1.2.2 Pitch rate

The pitch rate q is defined in terms of the load factor η in (2.12) and the (time derivative of the) normalized gravity/thrust force (acceleration) component $F_\alpha^{(gt)}$ in (2.13) using the approximation

$$q = -\frac{g}{Z_\alpha}\big(\dot{\eta} + \frac{\dot{F}_\alpha^{(gt)}}{g}\big) + \frac{g}{V}\eta \tag{2.16}$$

where Z_α is defined in terms of the coefficient C_Z for the aerodynamic force in the z-direction as[3]

$$Z_\alpha = \frac{1}{2m}\rho V^2 S_{ref}\frac{\partial C_Z}{\partial \alpha}|_{\alpha=0}$$

However, for small angles of attack α we have approximately

$$\frac{\partial C_Z}{\partial \alpha} = -\frac{\partial C_L}{\partial \alpha}$$

and hence we have approximately

$$Z_\alpha = -\frac{1}{2m}\rho V^2 S_{ref}\frac{\partial C_L}{\partial \alpha}|_{\alpha=0}. \tag{2.17}$$

[2]In actual numeric simulation this can be done by most easily in an approximate manner by using the values of α previous time steps to update η. From this one can then compute a new value for $\eta^{(a)}$ and subsequently also a new value for α.

[3]For small angles of attack we have (cf. (2.4)) approximately $\partial C_Z/\partial\alpha = -\partial C_L/\partial\alpha$.

If the angle of attack and thrust are assumed to be more slowly varying than the orientation of the aircraft the time derivative $\dot{F}_\alpha^{(gt)}$ can be computed using the approximation

$$\dot{F}_\alpha^{(gt)} = \frac{1}{m}\big(\dot{f}_z^{(g)}\cos(\alpha) - \dot{f}_x^{(g)}\sin(\alpha)\big) \tag{2.18}$$

and the relation

$$\dot{\boldsymbol{f}}^{(g)} = -\boldsymbol{\omega} \times \boldsymbol{f}^{(g)}, \tag{2.19}$$

where $\boldsymbol{f}^{(g)} = [f_x^{(g)}, f_y^{(g)}, f_z^{(g)}]^T$.

2.1.3 Roll channel

The roll channel is simply modeled as

$$\dot{p}^{(W)} = \frac{1}{\tau_p}(p_c^{(W)} - p^{(W)}), \tag{2.20}$$

where τ_p is the roll axis time constant, as dictated by the bare airframe characteristics or handling qualities, and $p_c^{(W)}$ is the commanded velocity axis roll rate.

2.1.4 Yaw channel

The yaw channel has no dynamics, only an algebraic condition which is needed in order to keep $\beta = 0$ at all times, viz.

$$\begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{bmatrix} \begin{bmatrix} p^{(W)} \\ -\frac{F_\beta}{V} \end{bmatrix}, \tag{2.21}$$

where

$$F_\beta = \frac{f_y^{(g)}}{m}. \tag{2.22}$$

2.1.5 Velocity

The dynamics for the airspeed are given by

$$\dot{V} = \frac{1}{m}\big((f_x^{(g)} + f_x^{(t)})\cos(\alpha) + f_z^{(g)}\sin(\alpha) - f_D\big) \tag{2.23}$$

where the drag force f_D and thrust force $f_x^{(t)}$ are modeled as in (2.9) and (2.11), respectively.

2.1.6 Computational dependencies

The main computational dependencies for computation of the velocity $\boldsymbol{V}$ in E and the quaternion $\boldsymbol{q}$ (or Euler angles) defining the relative orientation between B and E are outlined in Figure 2.1 below.

2.1.7 Summary of assumptions

The main assumptions used in the derivation below of the model in Section 2.1 are listed in Table 2.1. Note that they are of varying importance for the resulting closed loop model.

$$\left.\begin{array}{l} \left.\begin{array}{cccc} & t_{ss} & \rightarrow & V \\ & & & \beta\,(=0) \\ \eta_c \;\rightarrow & \begin{bmatrix}\eta\\ \dot\eta\end{bmatrix} & \rightarrow & \alpha \end{array}\right\} \rightarrow \boldsymbol{v} \\ \qquad\qquad\qquad\qquad \searrow \\ \left.\begin{array}{cccc} & & & q \\ p_c^{(W)} \;\rightarrow & p^{(W)} & \rightarrow & p \\ & & \searrow & \\ & & & r\end{array}\right\} \rightarrow \boldsymbol{q} \end{array}\right\} \rightarrow \begin{bmatrix}\boldsymbol{V}\\ \boldsymbol{q}\end{bmatrix}$$

Figure 2.1: Illustration of the main computational dependencies in the aircraft model.

Main assumptions		
No.	Assumption	Page
1	No force contribution from control surfaces.	15
2	Sideslip angle $\beta = 0$ at all times.	19
3	Left-right symm. airframe (aerodyn. y-force is 0 for $\beta = 0$).	19
4	No thrust force contribution in side force.	19
5	No thrust force in z-axis (cf. 4 above).	20
6	Ideal (first order) closed loop roll rate dynamics.	22
7	No inertia coupling into open loop pitch rate dynamics.	22
8	Aerodynamic z-force only dependent on α, and aerodynamic pitch mom. only dependent on α, q & control surf. settings.	22
9	Geometric nonlinearities in pitch linearized around $\alpha = 0$.	23
10	Airspeed is slowly varying in pitch dynamics.	25
11	Forces and moments linearized at $\alpha = q = 0$ in pitch dyn.	25
12	Aerodyn. force & mom. equilibr. for $\alpha = q = 0$ in pitch dyn.	25
13	Ideal synthesized closed loop pitch dynamics.	28
14	Orientation can change faster than angle of attack and thrust	31

Table 2.1: Assumptions used in the derivation of the model in Section 2.1.

2.2 Autopilot

We assume that the Earth fixed frame E is (right handed) Cartesian. The autopilot is capable of aligning the velocity vector $\boldsymbol{V}$ in E if the aircraft with a constant reference velocity vector $\boldsymbol{V}_r$. It consists of two subsystems; the guidance law and the command generators for roll and pitch commands. Additionally, a simple (e.g. PID-type) controller can be applied to control the magnitude of the velocity so that also the aircraft velocity magnitude converges to the magnitude of the reference velocity vector.

2.2.1 Guidance law

Let $\boldsymbol{R}(\boldsymbol{q})$ be the rotation matrix that relates quantities in B and E, respectively, so that for example

$$\boldsymbol{V} = \boldsymbol{R}(\boldsymbol{q})\boldsymbol{v}.$$

The acceleration commanded by guidance $\dot{\boldsymbol{v}}_g$ in B of the aircraft is computed from

$$\dot{\boldsymbol{v}}_g = \boldsymbol{R}(\boldsymbol{q})^T \boldsymbol{G}(\boldsymbol{V}, \boldsymbol{V}_r), \tag{2.24}$$

with

$$\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{V}_r) = \frac{c_g}{\|\boldsymbol{V}\|^2 \|\boldsymbol{V}_r\|} (\boldsymbol{\Omega}_g \times \boldsymbol{V}) \tag{2.25}$$

where $c_g > 0$ is a constant and

$$\boldsymbol{\Omega}_g = \boldsymbol{V} \times \boldsymbol{V}_r. \tag{2.26}$$

The vector $\boldsymbol{V}$ is aligned with $\boldsymbol{V}_r$ if and only if $\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{V}_r) = \boldsymbol{0}$ (for nonzero $\boldsymbol{V}, \boldsymbol{V}_r$), i.e. if and only if $\dot{\boldsymbol{v}}_g = \boldsymbol{0}$. The number

$$\frac{2V}{c_g}$$

can be interpreted as the time constant which describes the guidance law response $\dot{\boldsymbol{v}}_g$ to a stepwise change in the reference velocity $\boldsymbol{V}_r$. It is reasonable to make the dynamics of the guidance law (2.24) and (2.25) somewhat slower than the dynamics of the aircraft.

2.2.2 Orientation command generator

The command generator subsystem of the autopilot issues commands for roll and pitch in order to drive $\dot{\boldsymbol{v}}_g$ in (2.24) to zero.

Define the vector $\boldsymbol{\sigma} \in \mathbb{R}^3$ by

$$\boldsymbol{\sigma} = [\sin(\alpha), 0, -\cos(\alpha)]^T, \tag{2.27}$$

The command generator control law is given by

$$p_c^{(W)} = g\frac{c_g}{V}(\gamma_1 \dot{v}_{g,2} + \gamma_2 \Phi)\sqrt{1 - \frac{(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g)^2 + (\dot{v}_{g,2})^2}{c_g^2}}, \tag{2.28}$$

$$\eta_c = \delta \boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \tag{2.29}$$

where $\dot{v}_{g,2} = \boldsymbol{e}_2^T \dot{\boldsymbol{v}}_g$ and η_c is the commanded load factor and $\delta > 0$, $\gamma_1 > 0$, and $\gamma_2 < 0$. The constant δ defines the time constant

$$\frac{V}{g c_g \delta}$$

of the load factor tracking (with respect to the guidance law (2.24), (2.25)) and the constants γ_1, γ_2 define the dynamics for the roll-bank tracking adjustments and the relative influence of $\dot{v}_{g,2}$ and Φ for these (see (6.41)–(6.43) below).

3 Rigid Body Mechanics

The most fundamental assumption underlying the derivation of the model is that the dynamics of the aircraft can be described by rigid body motion.[1] To describe the dynamics we use the Newton-Euler (NE) equations and everything that follows will be the result of specifications, transformations and simplifications relating to these equations. We assume that there is a body fixed Cartesian frame B defined for the aircraft, with the orientation of the axes which is standard in flight mechanics, and that the origin of B is at the center of mass (CoM).

3.1 The Newton-Euler equations

The NE equations[2] for the motion in B for the CoM and the motion around the CoM of the aircraft read (Stevens & Lewis, 2003)

$$\begin{aligned} \dot{\boldsymbol{v}} &= \frac{1}{m}\boldsymbol{f} - \boldsymbol{\omega} \times \boldsymbol{v}, && (3.1) \\ \dot{\boldsymbol{\omega}} &= \boldsymbol{J}^{-1}(\boldsymbol{m} - \boldsymbol{\omega} \times \boldsymbol{J}\boldsymbol{\omega}), && (3.2) \end{aligned}$$

where $\boldsymbol{v} = [u, v, w]^T \in \mathbb{R}^3$ is the velocity, $\boldsymbol{\omega} = [p, q, r]^T \in \mathbb{R}^3$ is the angular velocity, $\boldsymbol{f} = [f_x, f_y, f_z]^T \in \mathbb{R}^3$ is the force, $\boldsymbol{m} \in \mathbb{R}^3$ is the moment, $m > 0$ is the mass and $\boldsymbol{J} \in \mathbb{R}^{3\times 3}$ is the moment of inertia matrix which we assume[3] has the form

$$\boldsymbol{J} = \begin{bmatrix} J_{xx} & 0 & J_{xz} \\ 0 & J_{yy} & 0 \\ J_{xz} & 0 & J_{zz} \end{bmatrix}.$$

The force vector $\boldsymbol{f}$ is made up of aerodynamic forces (which are mainly due to the orientation of the velocity vector $\boldsymbol{v}$ in B, on control surface settings and on $\boldsymbol{\omega}$, for a given Mach number and air density) and gravity and thrust. It will later turn out to be convenient to divide the forces f_x, f_y, f_z into aerodynamic components $f_x^{(a)}, f_y^{(a)}, f_z^{(a)}$, gravity induced components $f_x^{(g)}, f_y^{(g)}, f_z^{(g)}$ and thrust components $f_x^{(t)}, f_y^{(t)}, f_z^{(t)}$, respectively, so that

$$\begin{aligned} f_x &= f_x^{(a)} + f_x^{(g)} + f_x^{(t)}, \\ f_y &= f_y^{(a)} + f_y^{(g)} + f_y^{(t)}, \\ f_z &= f_z^{(a)} + f_z^{(g)} + f_z^{(t)}. \end{aligned} \qquad (3.3)$$

The moment vector $\boldsymbol{m}$ consists of aerodynamic moments.[4] Aircraft are normally controlled using control surfaces located forward (canards) or aft (elevons, elevators, ailerons and tail fin) of the CoM, and in both these cases control surface deflections in general give considerably larger relative change in $\boldsymbol{m}$ than in $\boldsymbol{f}$. It is therefore reasonable, as an approximation, to neglect the Assumption 1.

[1] This means in particular that we assume that the total mass and mass distribution are constant (and all mass flow effects are neglected).

[2] We assume that the force and moment terms are at least piecewise continuous functions so that a unique solution to the initial value problem always exists, at least locally in time.

[3] If the mass distribution is symmetric when mirrored in the xz-plane we have $J_{xy} = J_{yx} = J_{yz} = J_{zy} = 0$, which is a common assumption common in flight mechanics when the coordinate axes in B have the standard orientation Stevens & Lewis (2003).

[4] Thus, we assume that thrust gives no moment contribution (in particular we do not consider thrust vectoring) but extension to this case is trivial.

force contributions from the control surfaces and we shall do so here. Further, it is convenient to partition the moment $\boldsymbol{m}$ as $\boldsymbol{m} = \boldsymbol{m}^{(a)} + \boldsymbol{u}$, where $\boldsymbol{u} = [u_x, u_y, u_z]^T \in \mathbb{R}^3$ is the moment caused by the control surface deflections (from their nominal position) and represents the *control variable*, and $\boldsymbol{m}^{(a)} = [m_x^{(a)}, m_y^{(a)}, m_z^{(a)}]^T$ is the remaining aerodynamic moment (which is mainly due to the orientation of the velocity vector $\boldsymbol{v}$ in B and on $\boldsymbol{\omega}$, for a given Mach number and air density).

3.2 Motion in an Earth fixed frame

To get the complete motion in an Earth fixed Cartesian frame E (assumed inertial) it is necessary to complement (3.1), (3.2) with kinematic and dynamic relations which translate the motion to E, e.g. (Stevens & Lewis, 2003)

$$\begin{aligned} \boldsymbol{V} &= \boldsymbol{R}(\boldsymbol{q})\boldsymbol{v}, \\ \dot{\boldsymbol{q}} &= \frac{1}{2}\boldsymbol{q} \circ (0, \boldsymbol{\omega}), \end{aligned} \tag{3.4}$$

where $\boldsymbol{V} \in \mathbb{R}^3$ is the velocity for the CoM expressed in E and $\boldsymbol{R}(\boldsymbol{q}) \in \mathbb{R}^{3\times3}$ is the rotation matrix which relates B and E. The rotation matrix $\boldsymbol{R}(\boldsymbol{q})$ is here expressed as a function of an orientation quaternion $\boldsymbol{q} \in \mathbb{H}$ (the symbol $\circ$ denotes quaternion multiplication and $(0, \boldsymbol{\omega})$ is the purely imaginary quaternion obtained from the vector $\boldsymbol{\omega}$). Finally, the position in E is obtained by integrating $\boldsymbol{V}$ over time.

For later use we note that Newton's equation for the motion of the CoM in E reads

$$\dot{\boldsymbol{V}} = \frac{1}{m}\boldsymbol{F} \tag{3.5}$$

where $\boldsymbol{F}$ is the force vector in E, viz.,

$$\boldsymbol{F} = \boldsymbol{R}(\boldsymbol{q})\boldsymbol{f} \tag{3.6}$$

and from the theory of relative motion (cf. Appendix A) we have

$$\boldsymbol{R}(\boldsymbol{q})^T\dot{\boldsymbol{V}} = \dot{\boldsymbol{v}} + \boldsymbol{\omega} \times \boldsymbol{v} \tag{3.7}$$

(which of course together yields the force equation (3.1) in B).

3.3 Aerodynamic coordinates

In aerodynamic contexts it is common to use other coordinates than the Cartesian defined in B, in particular it is common to employ angle of attack, sideslip angle and airspeed (or total velocity). The main reason for introducing these (spherical) coordinates is that the most important force (at least for an airfoil), the lift, is essentially linear (or affine) in α over a large interval. We therefore introduce the angle of attack α, sideslip angle[5] β and airspeed V as[6]

$$\begin{aligned} \alpha &= \arctan(w/u), \\ \beta &= \arcsin(v/V), \\ V &= \|\boldsymbol{v}\| = \sqrt{u^2 + v^2 + w^2}, \end{aligned}$$

[5] The definition of β used here is the one most often used in aircraft contexts. In missile contexts the definition of β is usually taken as $\beta = \arctan(v/u)$. In our analysis, which is based on linearization, the end result (the simplified model) will the same regardless of which definition is used.

[6] Since we shall not consider aircraft that fly "sideways" or "backwards" there is no practical restriction in making the domain of definition for α, β as "small" as we do.

where $\alpha, \beta \in (-\pi/2, \pi/2)$ and $V > 0$, which gives us the inverse relations

$$\begin{aligned} u &= V\cos(\alpha)\cos(\beta), \\ v &= V\sin(\beta), \\ w &= V\sin(\alpha)\cos(\beta). \end{aligned}$$

In terms of these variables, the force equation (3.1) can be expressed with the following three equations (Stevens & Lewis, 2003; Johansson, 1998)

$$\dot{\alpha} = \frac{F_\alpha}{V} - p\cos(\alpha)\tan(\beta) - r\sin(\alpha)\tan(\beta) + q, \tag{3.8}$$

$$\dot{\beta} = \frac{F_\beta}{V} + p\sin(\alpha) - r\cos(\alpha), \tag{3.9}$$

$$\dot{V} = F_V, \tag{3.10}$$

where

$$F_\alpha = \frac{f_z\cos(\alpha) - f_x\sin(\alpha)}{m\cos(\beta)}, \tag{3.11}$$

$$F_\beta = \frac{f_y\cos(\beta) - f_x\cos(\alpha)\sin(\beta) - f_z\sin(\alpha)\sin(\beta)}{m}, \tag{3.12}$$

$$F_V = \frac{f_y\sin(\beta) + f_x\cos(\alpha)\cos(\beta) + f_z\sin(\alpha)\cos(\beta)}{m}. \tag{3.13}$$

In the following chapters, the system (3.2) and (3.8)–(3.10) will, through a sequence of steps, be simplified to yield the equations in the model in Section 2.1.

3.3.1 Wind axes

If we define the two vectors $\boldsymbol{\lambda}, \boldsymbol{\sigma} \in \mathbb{R}^3$ by

$$\boldsymbol{\lambda} = [-\cos(\alpha)\tan(\beta), 1, -\sin(\alpha)\tan(\beta)]^T, \tag{3.14}$$

$$\boldsymbol{\sigma} = [\sin(\alpha), 0, -\cos(\alpha)]^T, \tag{3.15}$$

and recall that

$$\boldsymbol{v} = V[\cos(\alpha)\cos(\beta), \sin(\beta), \sin(\alpha)\cos(\beta)]^T \tag{3.16}$$

we see that the quantities in (3.8)–(3.13) can be expressed in terms of (scaled) projections onto $\boldsymbol{\lambda}, \boldsymbol{\sigma}$ and $\boldsymbol{v}$. These three vectors are mutually orthogonal and $[\boldsymbol{v}], [\boldsymbol{\lambda}], [\boldsymbol{\sigma}]$ represent the so called *wind axes*. The unit vectors $\boldsymbol{e_v}, \boldsymbol{e_\lambda}, \boldsymbol{\sigma}$ define a (left-handed) basis W for B which is often more convenient for representation of aerodynamic forces than the standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. In particular, if we set $\beta = 0$ we can define the pitch plane lift[7] and drag forces $f_L, f_D \in \mathbb{R}$ as

$$f_L = f_x^{(a)}\sin(\alpha) - f_z^{(a)}\cos(\alpha), \tag{3.17}$$

$$f_D = -f_x^{(a)}\cos(\alpha) - f_z^{(a)}\sin(\alpha). \tag{3.18}$$

The pitch plane lift and drag forces[8] f_L, f_D are the components along $\boldsymbol{\sigma}$ and $-\boldsymbol{e_v}$, respectively, of the aerodynamic force $\boldsymbol{f}^{(a)}$ (when $\beta = 0$ and $f_y^{(a)} = 0$) which is a case that will be of interest below.

[7] As alluded to above, f_L is exactly linear (affine) in α according to e.g. thin airfoil theory.

[8] Lift and drag are defined in terms of the wind axes in general but we shall here only need to consider the (pitch plane) case obtained when $\beta = 0$.

Another important aspect of the wind axes vectors $\boldsymbol{e_v}, \boldsymbol{e_\lambda}, \boldsymbol{\sigma}$ is that they also represent a natural basis for the angular velocities of the aircraft. Thus, $\boldsymbol{\omega}$ can be represented as

$$\boldsymbol{\omega} = p^{(W)}\boldsymbol{e_v} + q^{(W)}\boldsymbol{e_\lambda} + r^{(W)}\boldsymbol{\sigma}$$

where

$$p^{(W)} = \boldsymbol{e_v}^T\boldsymbol{\omega}, \quad q^{(W)} = \boldsymbol{e_\lambda}^T\boldsymbol{\omega}, \quad r^{(W)} = \boldsymbol{\sigma}^T\boldsymbol{\omega}. \tag{3.19}$$

By setting $\beta = 0$ in (3.14)–(3.16) it is clear that an aircraft in coordinated flight ("bank-to-turn flight") should always (ideally) be maneuvered by a combination of rotations around the body y-axis, i.e. the axis $[\boldsymbol{e_\lambda}]$ (pitching maneuver), and rotations around velocity vector axis, i.e. the axis $[\boldsymbol{e_v}]$ (rolling maneuver). This ideal decomposition of the rotational motion is not always possible to achieve, see below, but it can serve as a template (and we shall assume that the aircraft is indeed capable of maneuvering like this).

3.4 Equilibrium points

In flight mechanical studies of maneuvering aircraft it is common to neglect gravity and assume that the airspeed (and air density) is constant (Goman & Khramstovsky, 1997; Jahnke, 1998; Goman *et al.*, 2007). This simplifies the analysis and does not in general change the qualitative characteristics of problems related to maneuvering in a significant way. The focus will then be on the dynamics in the part of the NE equations represented by (3.2) and (3.8)–(3.10). For these equations, the most fundamental problem is to determine the existence (and nature of) equilibrium points (Goman *et al.*, 2007) since these points represent sustainable flight conditions (under the simplifying assumptions).

As noted above, one of the natural basic motions of an aircraft is rotation around the velocity vector axis $[\boldsymbol{e_v}]$. However, it is not in general possible (due to properties of the airframe and control surfaces) to specify $\alpha, \beta, p^{(W)}$, have $q^{(W)} = r^{(W)} = 0$ and obtain equilibrium in (3.2) and (3.8)–(3.10) (Goman *et al.*, 2007). Thus, the quantities $q^{(W)}, r^{(W)}$ have to be considered as free variables and be allowed to to take arbitrary nonzero values when seeking equilibrium to (3.2) and (3.8)–(3.10). Still, it is very natural to use $\alpha, \beta, p^{(W)}$ as *controlled variables* (ones for which desired values are commanded during maneuvering) and this approach is (essentially) the one we shall take.

4 Simplifying the Equations I

In this chapter we perform simplifications which can be motivated by considering only the most important functional dependencies in various quantities and by considering the relative sizes of various terms occurring in the equations. Moreover, we shall linearize the geometric nonlinearities (those related to the change to aerodynamic coordinates α, β, V).

4.1 Simplified force equation

Assumption 2.

Since we shall focus on the case of coordinated flight, i.e. flight where the side slip angle β is zero at all times, we shall start by investigating what consequences such a condition has.

4.1.1 Sideslip angle

From (3.9) we see that β is identically 0 over a time interval $[t_0, t_1]$ if and only if $\beta(t_0) = 0$ and the relation

$$0 = \frac{F_\beta}{V} + p\sin(\alpha) - r\cos(\alpha) \tag{4.1}$$

holds over $[t_0, t_1]$. We may therefore we use (4.1) together with the assumption $\beta(t_0) = 0$ for some t_0 as a necessary and sufficient condition for $\beta \equiv 0$. In fact, we may then assume[1] that $F_\beta = f_y/m$ since this is what the expression for F_β in (3.12) reduces to when $\beta = 0$. Moreover, for an airframe which is symmetric when mirrored in the x, z-plane in B the aerodynamic force component $f_y^{(a)}$ is an odd (continuous) function of β and therefore must vanish when $\beta = 0$. If we restrict attention to such airframes and assume that the trust contribution $f_y^{(t)}$ in (3.3) is zero we thus have

Assumption 3.

Assumption 4.

$$F_\beta = \frac{f_y^{(g)}}{m}. \tag{4.2}$$

This expression for F_β is used in the model in Section 2.1. We also note that $\beta \equiv 0$ is equivalent to $v \equiv 0$ which in its turn implies $\dot{\boldsymbol{v}} \in [\boldsymbol{e_v}, \boldsymbol{\sigma}] = [\boldsymbol{e}_2]^\perp$ for all times and we note that (4.1) and (4.2) together yield

$$r^{(W)} = \boldsymbol{\omega}^T\boldsymbol{\sigma} = -\frac{f_y^{(g)}}{mV} \tag{4.3}$$

which will be used below.

4.1.2 Angle of attack

If we use the assumption $\beta = 0$ in (3.8) we obtain

$$\dot{\alpha} = \frac{F_\alpha}{V} + q, \tag{4.4}$$

where F_α from (3.11) here becomes

$$F_\alpha = \frac{f_z\cos(\alpha) - f_x\sin(\alpha)}{m}. \tag{4.5}$$

[1] Assume $\beta(t_0) = 0$ and $f_y/(mV) + p\sin(\alpha) - r\cos(\alpha) = 0$ over a time interval $\mathcal{I}$ containing t_0 and that f_x, f_y, f_z are C^1-functions there. Then by (3.9), (3.12) we have $\dot{\beta} = (1/(mV))(f_y(\cos(\beta) - 1) - f_x\cos(\alpha)\sin(\beta) - f_z\sin(\alpha)\sin(\beta))$ over $\mathcal{I}$ and by uniqueness of solutions $\beta \equiv 0$ is the only solution to this differential equation.

With the definitions in (3.3) we can also divide F_α correspondingly as

$$F_\alpha = F_\alpha^{(a)} + F_\alpha^{(gt)}, \tag{4.6}$$

where the aerodynamic component $F_\alpha^{(a)}$ can be expressed in terms of the pitch plane lift force f_L in (3.17) as

$$F_\alpha^{(a)} = \frac{f_z^{(a)}\cos(\alpha) - f_x^{(a)}\sin(\alpha)}{m} = -\frac{f_L}{m}, \tag{4.7}$$

and the gravity/thrust component $F_\alpha^{(gt)}$ defined by

$$F_\alpha^{(gt)} = \frac{1}{m}\big(f_z^{(g)}\cos(\alpha) - (f_x^{(g)} + f_x^{(t)})\sin(\alpha)\big). \tag{4.8}$$

Assumption 5.

Here, we have also introduced the assumption that $f_z^{(t)} = 0$, which together with the previous assumption that $f_y^{(t)} = 0$ means that thrust only acts along the x-axis in B.

The expression for $F_\alpha^{(gt)}$ in (4.8) is used in the model in Section 2.1 (for some computational aspects on this term, see Sec. 5.4.3 below).

4.1.3 Velocity

If we use the assumption $\beta = 0$ in (3.10) we obtain

$$\dot{V} = F_V = \frac{f_x\cos(\alpha) + f_z\sin(\alpha)}{m}. \tag{4.9}$$

With the definitions in (3.3) we can also split F_V as

$$F_V = F_V^{(a)} + F_V^{(gt)}, \tag{4.10}$$

where the aerodynamic component $F_V^{(a)}$ is related to the pitch plane drag force f_D in (3.18) as

$$F_V^{(a)} = -\frac{f_D}{m} \tag{4.11}$$

and the gravity/thrust component $F_V^{(gt)}$ is defined by

$$F_V^{(gt)} = \frac{1}{m}\big((f_x^{(g)} + f_x^{(t)})\cos(\alpha) + f_z^{(g)}\sin(\alpha)\big). \tag{4.12}$$

Summing up, equation (4.4) together with the condition $\beta = 0$ and (4.9) yield a complete description of the evolution of the variables α, β and V in the transformed and simplified version of the force equation (3.8)–(3.10), provided we specify either time evolutions or functional relations for the force components $f_x^{(a)}, f_x^{(g)}, f_x^{(t)}, f_y^{(g)}, f_z^{(a)}, f_z^{(g)}$. (Recall that the aerodynamic forces have implicit dependence on variables other than those in the NE equations, or their transformed counterparts, such as dependence on air density which is mostly dependent on altitude.)

4.2 Simplified moment equation

We consider here aircraft that have "conventional" airframes with certain symmetries in shape and mass distribution. Therefore we can assume a certain form of the moment of inertia matrix and certain relations between the elements in it to hold.

4.2.1 Roll channel

The open loop dynamics for p can be extracted form (3.2) as

$$\begin{aligned}\dot{p} = J_{zz}\frac{qrJ_{yy} - q(pJ_{xz} + rJ_{zz})}{J_{xx}J_{zz} - J_{xz}^2} + J_{xz}\frac{pqJ_{yy} - q(rJ_{xz} + pJ_{xx})}{J_{xx}J_{zz} - J_{xz}^2} \\ + \frac{J_{zz}}{J_{xx}J_{zz} - J_{xz}^2}(m_x^{(a)} + u_x) - \frac{J_{xz}}{J_{xx}J_{zz} - J_{xz}^2}(m_z^{(a)} + u_z). \quad (4.13)\end{aligned}$$

For a fighter aircraft[2] the ratio J_{zz}/J_{xz} can be in the order of 30 or more and since $m_z^{(a)} + u_z$ will be small compared to $m_x^{(a)} + u_x$, at least during maneuvering (with small β), the last moment term on the right in (4.13) is likely to be small compared the one before it. Moreover, since the mass distribution of the aircraft can generally be assumed to be well known the first two terms in (4.13) can be canceled, at least approximately, by adding a term to the control law. Together this means that it is reasonable to assume that the control system on the aircraft can synthesize a closed loop roll response as[3]

$$\dot{p} = \frac{1}{\tau_p}(p_c - p), \quad (4.14)$$

where p_c is the commanded value for the roll rate around the y-axis in B and τ_p is the associated time constant, as dictated by standard handling qualities requirements (Hodgkinson, 1999, p. 94,133).

4.2.1.1 Velocity axis roll

As indicated before, we shall take $p^{(W)}$ in (3.19) to be the controlled variable for roll motion and from the definition (3.19) we obtain for the case of coordinated flight ($\beta = 0$)

$$\dot{p}^{(W)} = \dot{p}\cos(\alpha) + \dot{r}\sin(\alpha) + \dot{\alpha}(r\cos(\alpha) - p\sin(\alpha)). \quad (4.15)$$

The open loop dynamics for r can be extracted from (3.2) as

$$\begin{aligned}\dot{r} = J_{zx}\frac{q(pJ_{xz} + rJ_{zz}) - qrJ_{yy}}{J_{xx}J_{zz} - J_{xz}^2} + J_{xx}\frac{q(pJ_{xx} + rJ_{xz}) - pqJ_{yy}}{J_{xx}J_{zz} - J_{xz}^2} \\ - \frac{J_{xz}}{J_{xx}J_{zz} - J_{xz}^2}(m_x^{(a)} + u_x) + \frac{J_{xx}}{J_{xx}J_{zz} - J_{xz}^2}(m_z^{(a)} + u_z) \quad (4.16)\end{aligned}$$

and by combining (4.13), (4.15) and (4.16) it follows that

$$\begin{aligned}\dot{p}^{(W)} = (J_{zz}\cos(\alpha) - J_{xz}\sin(\alpha))\frac{qrJ_{yy} - q(pJ_{xz} + rJ_{zz})}{J_{xx}J_{zz} - J_{xz}^2} \\ + (J_{xz}\cos(\alpha) - J_{xx}\sin(\alpha))J_{xz}\frac{pqJ_{yy} - q(rJ_{xz} + pJ_{xx})}{J_{xx}J_{zz} - J_{xz}^2} \\ + (J_{zz}\cos(\alpha) - J_{xz}\sin(\alpha))\frac{J_{zz}}{J_{xx}J_{zz} - J_{xz}^2}(m_x^{(a)} + u_x) \\ - (J_{xz}\cos(\alpha) - J_{xx}\sin(\alpha))\frac{J_{xz}}{J_{xx}J_{zz} - J_{xz}^2}(m_z^{(a)} + u_z) \\ + \dot{\alpha}(r\cos(\alpha) - p\sin(\alpha)).\end{aligned}$$

[2] For e.g. the ADMIRE model Forssell & Nilsson (2005) the values are $J_{xx} = 21000\text{kg/m}^2$, $J_{yy} = 81000\text{kg/m}^2$, $J_{zz} = 101000\text{kg/m}^2$ and $J_{xz} = -2500\text{kg/m}^2$.

[3] A first order form for the open loop response is a reasonable approximation for many aircraft due to the shape of the aerodynamic roll damping moment Hodgkinson (1999) and (presumably) therefore this form of the response is kept in handling qualities requirements.

By the same argumentation that was used above it is clear that the term describing moment around the z-axis is in general likely to be considerably smaller than the one describing moment around the x-axis, and the remaining terms contain known or measurable quantities which can, at least approximately, be canceled by the control law. Hence, it is reasonable to assume that also the closed loop roll response around the velocity vector can be synthesized to have the form

Assumption 6.

$$\dot{p}^{(W)} = \frac{1}{\tau_p}(p_c^{(W)} - p^{(W)}), \tag{4.17}$$

where $p_c^{(W)}$ is the commanded value for the velocity axis roll rate $p^{(W)}$ in (3.19).

4.2.2 Pitch channel

For the pitch channel finally we extract open loop dynamics for q from (3.2) as

$$\dot{q} = -\frac{r(pJ_{xx} + rJ_{xz})}{J_{yy}} - \frac{p(pJ_{xz} + rJ_{zz})}{J_{yy}} + \frac{1}{J_{yy}}(m_y^{(a)} + u_y) \tag{4.18}$$

With similar argumentation as for the roll channel about the possibility for the control law to, at least approximately, cancel (smaller) terms we see that it is reasonable to approximate (or model) it as

Assumption 7.

$$\dot{q} = \frac{1}{J_{yy}}(m_y^{(a)} + u_y). \tag{4.19}$$

Thus, we shall use (4.19) as our model of the open loop moment equation for the pitch channel which will later lead to a closed loop model for the pitch channel (when the α, q dynamics have been further developed).

4.2.3 Yaw channel

If we use p to describe the roll motion, as in (4.14), we can simply postulate p and then (at each time instant) solve for r in (4.1) (with (4.2)) to satisfy the sufficient condition for β to remain zero at all times (i.e start with $\beta(t_0) = 0$ at some t_0 and then satisfy (4.1) for all times onward). However, when using $p^{(W)}$ in (3.19) instead to describe the roll motion, as in (4.17), it is clear that we at each time instant must satisfy the system

$$\begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{bmatrix} \begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} p^{(W)} \\ -\frac{F_\beta}{V} \end{bmatrix}. \tag{4.20}$$

Since the matrix on the left always has full rank (indeed, it is orthogonal) this will define (p, r) uniquely in such a way that both conditions (4.17) and (4.1) are fulfilled. For easy reference we write down the solution to (4.20) which is

$$\begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{bmatrix} \begin{bmatrix} p^{(W)} \\ -\frac{F_\beta}{V} \end{bmatrix}. \tag{4.21}$$

4.3 Simplified nonlinear pitch plane model

Additional simplifications over those given in previous sections can be obtained if one observes the main functional dependencies in $f_z^{(a)}$ and $m_y^{(a)}$. The aerodynamic force $f_z^{(a)}$ generally mainly depends on α (more than on q for fixed V) (Stevens & Lewis, 2003, p. 76) and the moment $m_y^{(a)}$ depends mainly on α and q. We are therefore going to assume that these are the only functional dependencies in $f_z^{(a)}$ and $m_y^{(a)}$.

Assumption 8.

Further, the relative sizes of the geometric nonlinearities (the trigonometric functions) occurring in Sec. 4.1 will in general be determined by the simple fact that α will be small or at most moderate.[4] Thus, it is reasonable to replace the geometric nonlinearities with their first order approximation (linearization) around $\alpha = 0$. Assumption 9.

If we use these assumptions and approximations in (4.4) and (4.19) we obtain a model of the form

$$\begin{aligned} \dot{\alpha} &= \frac{Z(\alpha)}{V} + q + \frac{F_\alpha^{(gt)}}{V}, && (4.22) \\ \dot{q} &= M(\alpha, q) + U^{(y)}, && (4.23) \end{aligned}$$

(with $F_\alpha^{(gt)}$ as in (4.8)) where we have introduced the normalized forces[5] (accelerations) and normalized moments (angular accelerations) according to

$$Z = \frac{f_z^{(a)}}{m}, \quad M = \frac{m_y^{(a)}}{J_{yy}}, \quad U^{(y)} = \frac{u_y}{J_{yy}}. \tag{4.24}$$

This model will, after some further simplifications, be one of the core parts of the overall aircraft model. We note that the normalized force Z is in general almost linear (affine) in α (for small to moderate values, cf. (4.7)).

[4]Since we shall not consider high angle of attack flight it is reasonable to assume that α is at most of the order $25°$.

[5]Note that Z and M have implicit dependence on V and on air density/altitude.

5 Simplifying the Equations II

In this chapter we further simplify the equations of motion by linearizing also the aerodynamic force and moment terms. Since it is generally true for maneuvering aircraft that V evolves on a slower time scale than α, q we shall in the treatment of the pitch plane dynamics consider V as slowly varying (i.e. set $\dot{V} = 0$). In the last section we shall derive a dynamic model for the evolution of V. Assumption 10.

5.1 Linearized model of the pitch plane dynamics

If we linearize the aerodynamic forces and moments in the simplified pitch plane dynamics equations (4.22), (4.23) around a reference point $(\alpha, q) = (\alpha_0, q_0)$ where α_0, q_0 are small (so that (4.22), (4.23) provide a good approximation to the pitch dynamics) and slowly varying (so that $\dot{\alpha}_0 = 0, \dot{q}_0 = 0$) we obtain

$$\dot{\tilde{\alpha}} = \frac{Z_\alpha(\alpha_0)}{V}\tilde{\alpha} + \tilde{q} + R_\alpha(\alpha_0, q_0), \tag{5.1}$$

$$\dot{\tilde{q}} = M_\alpha(\alpha_0, q_0)\tilde{\alpha} + M_q(\alpha_0, q_0)\tilde{q} + \tilde{U} + R_q(\alpha_0, q_0), \tag{5.2}$$

where $U_0^{(y)}$ is a normalized reference moment and we have introduced the deviations

$$\tilde{\alpha} = \alpha - \alpha_0, \quad \tilde{q} = q - q_0, \quad \tilde{U} = U^{(y)} - U_0^{(y)}, \tag{5.3}$$

and notation

$$Z_\alpha = \frac{dZ}{d\alpha}, \quad M_\alpha = \frac{\partial M}{\partial \alpha}, \quad M_q = \frac{\partial M}{\partial q},$$

(with Z, M as in (4.24)) and

$$R_\alpha(\alpha_0, q_0) = \frac{Z(\alpha_0)}{V} + q_0 + \frac{F_\alpha^{(gt)}}{V}, \quad R_q(\alpha_0, q_0) = M(\alpha_0, q_0) + U_0^{(y)}, \tag{5.4}$$

(with $F_\alpha^{(gt)}$ as in (4.8)). One choice of the reference point (α_0, q_0) for the linearized system (5.1), (5.2) is the one where the two remainder terms $R_\alpha(\alpha_0, q_0)$ and $R_q(\alpha_0, q_0)$ in (5.4) become zero for $U_0^{(y)} = 0$. This would correspond to an equilibrium point for the linearized system with gravity and thrust included, but since this would depend on the orientation of the vehicle and throttle setting it will not turn out to be the most convenient for our future developments.

5.1.1 Aerodynamic equilibrium

The choice of linearization point greatly affects the dynamics in the linearized model[1] but since we here focus on the resulting closed loop behavior we use a "generic" reference point and choose Assumption 11.

$$(\alpha_0, q_0) = (0, 0).$$

We shall also assume that at the reference point $\alpha_0 = q_0 = 0$ the system (5.1), (5.2) is at equilibrium if $F_\alpha^{(gt)} = 0$ and $U_0^{(y)} = 0$. This will make the effects of gravity explicitly visible. It moreover implies that $Z(0) = 0$, i.e. the airframe provides zero lift at zero angle of attack, but this is reasonable for Assumption 12.

[1] Various aspects of the choice of linearization point and its consequences are discussed in (Robinson, 2010).

most airframes since the actual value is likely to be close to zero. Likewise the equilibrium condition implies $M(\alpha_0, q_0) = 0$, i.e. aerodynamic moment balance with control surfaces in neutral position, which can be made to hold by a redefinition of the moment terms. (Taken together this means that the reference point $\alpha_0 = q_0 = 0$ can be thought of as one which gives *aerodynamic equilibrium* to the system (5.2).)

5.1.2 Aerodynamic coefficients

The normalized forces and moments in (5.1), (5.2) are commonly described by aerodynamic coefficients (Stevens & Lewis, 2003; Raymer, 2006). For instance,

$$Z(\alpha) = \frac{1}{2m}\rho V^2 S_{ref} C_Z(\alpha) \tag{5.5}$$

where ρ is the air density, S_{ref} is the reference area and C_Z is the aerodynamic force coefficient in the (positive) z-direction in B. In particular we see that $Z(\alpha) = 0$ is equivalent to $C_Z(\alpha) = 0$ so the assumption above that $\alpha_0 = q_0 = 0$ yields aerodynamic equilibrium also implies $C_Z(0) = 0$.

The coefficient C_Z can be Taylor expanded around $\alpha = 0$ and the linear approximation of C_Z is then

$$C_Z(\alpha) = \frac{\partial C_Z}{\partial \alpha}|_{\alpha=0}\alpha \tag{5.6}$$

Usually, this approximation is accurate for α up to, at least, 25° and this is one of the reasons for the widespread use of linearized pitch plane models. In terms of the linear representation (5.6) we have

$$Z_\alpha(0) = \frac{1}{m}\frac{\partial f_z^{(a)}}{\partial \alpha}|_{\alpha=0} = \frac{1}{2m}\rho V^2 S_{ref}\frac{\partial C_Z}{\partial \alpha}|_{\alpha=0}. \tag{5.7}$$

For later use we note also that in terms of aerodynamic coefficients the definition (3.17) can be written

$$C_L(\alpha) = C_X(\alpha)\sin(\alpha) - C_Z(\alpha)\cos(\alpha),$$

where C_L is the lift force coefficient corresponding to the lift force f_L in (3.17) and C_X is the coefficient for the aerodynamic force in the x-direction in B. It follows that

$$\frac{\partial C_L}{\partial \alpha}|_{\alpha=0} = C_X(0) - \frac{\partial C_Z}{\partial \alpha}|_{\alpha=0}$$

but the term $C_X(0)$ is often[2] more than two orders of magnitude smaller than the other terms so it can be ignored. Thus, near $\alpha = 0$, the slope of the lift force coefficient and the slope of the z-force coefficient are, after a sign change, essentially identical.

The other linearized force and moment quantities in (5.1), (5.2) can analogously be described in terms of the appropriate aerodynamic force and moment coefficients and their derivatives.

5.2 The short period approximation

The equations for the linearized pitch plane dynamics (5.1), (5.2) around the reference point $(\alpha_0, q_0) = (0, 0)$ can be written on scalar (SISO) form in terms

[2] In the transonic region this is no longer true, but $C_X(0)$ is then still significantly smaller than the term $\partial C_Z/\partial\alpha|_{\alpha=0}$.

of the variable α by eliminating q as

$$\begin{aligned}
\ddot{\alpha} &= \frac{Z_\alpha}{V}\dot{\alpha} - \frac{Z_\alpha \dot{V}}{V^2}\alpha + M_\alpha \alpha + M_q q + \tilde{U} + \frac{d}{dt}\frac{F_\alpha^{(gt)}}{V} \\
&= \frac{Z_\alpha}{V}\dot{\alpha} - \frac{Z_\alpha \dot{V}}{V^2}\alpha + M_\alpha \alpha + M_q(\dot{\alpha} - \frac{Z_\alpha}{V}\alpha - \frac{F_\alpha^{(gt)}}{V}) \\
&\qquad + \tilde{U} + \frac{d}{dt}\frac{F_\alpha^{(gt)}}{V} \\
&= \big(\frac{Z_\alpha}{V} + M_q\big)\dot{\alpha} + \big(M_\alpha - M_q\frac{Z_\alpha}{V}\big)\alpha \\
&\qquad + \tilde{U} - M_q\frac{F_\alpha^{(gt)}}{V} + \frac{d}{dt}\frac{F_\alpha^{(gt)}}{V}, \qquad (5.8)
\end{aligned}$$

where

$$Z_\alpha = Z_\alpha(0), \quad M_\alpha = M_\alpha(0,0), \quad M_q = M_q(0,0)$$

and we have used the assumption $\dot{V} = 0$. The second order linear dynamics described by the differential equation (5.8) is often referred to as the *short period approximation*, Ananthkrishnan & Unnikrishnan (2001), of the pitch plane dynamics.

The short period approximation (5.8) can be parametrized as

$$\ddot{\alpha} + 2\zeta_{sp}\omega_{sp}\dot{\alpha} + \omega_{sp}^2\alpha = \tilde{U} + U^{(gt)}, \qquad (5.9)$$

where

$$\tilde{U} = U^{(y)} - U_0^{(y,a)}, \quad U^{(gt)} = -M_q\frac{F_\alpha^{(gt)}}{V} + \frac{\dot{F}_\alpha^{(gt)}}{V}, \qquad (5.10)$$

and the undamped natural frequency ω_{sp} and damping ζ_{sp}, respectively, are given by

$$\begin{aligned}
\omega_{sp}^2 &= -(M_\alpha - M_q Z_\alpha/V), \qquad (5.11)\\
2\zeta_{sp}\omega_{sp} &= -(M_q + Z_\alpha/V). \qquad (5.12)
\end{aligned}$$

The expressions (5.11), (5.12) for the undamped natural frequency and damping are the ones often given in the literature to describe the short period approximation[3] (cf. e.g. (Ananthkrishnan & Unnikrishnan, 2001, Eq. (10)), or (Stevens & Lewis, 2003, Eq. (4.2-10)) with $Z_{\dot{\alpha}}, Z_q, M_{\dot{\alpha}} = 0$).

5.2.1 State space representation

A state space representation of (5.9) will turn out to be useful below. The simplest choice of state variables x_1, x_2 is

$$x_1 = \alpha, \quad x_2 = \dot{x}_1$$

which gives the representation

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ M_\alpha - M_q\frac{Z_\alpha}{V} & \frac{Z_\alpha}{V} + M_q \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
&= \begin{bmatrix} 0 & 1 \\ -\omega_{sp}^2 & -2\zeta_{sp}\omega_{sp} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad (5.13)
\end{aligned}$$

[3]Usually, however, the linearization point (α_0, q_0) in (5.1), (5.2) is one which gives equilibrium straight and level horizontal flight (Ananthkrishnan & Unnikrishnan, 2001).

where the new control variable u is defined by

$$u = \tilde{U} - M_q \frac{F_\alpha^{(gt)}}{V} + \frac{\dot{F}_\alpha^{(gt)}}{V} = \tilde{U} + U^{(gt)} \tag{5.14}$$

and $\tilde{U}, U^{(gt)}$ are defined as in (5.10). We note in passing that the eigenvalues $\lambda_{1,2}$ of the matrix on the right in (5.13), which are the roots of the characteristic polynomial of the system (i.e. the poles to the transfer function in (5.9)), are given by

$$\lambda_{1,2} = -\omega_{sp}\big(\zeta_{sp} \mp i\sqrt{1-\zeta_{sp}^2}\big)$$

(assuming that $\omega_{sp} > 0$ and $\zeta_{sp} \in (0,1)$).

5.3 Closed Loop System

It is easy to see that the system (5.13) is controllable and therefore a linear state feedback controller can be added to the system to synthesize any closed loop poles. Indeed, if a state feedback of the form

$$\tilde{u} = [k_1\ k_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u$$

is added to the system (5.13) then the new system has the same form as the old (with new control input $\tilde{u}$), but where the substitutions

$$\begin{aligned} M_\alpha &\to M_\alpha + k_1, \\ M_q &\to M_q + k_2, \end{aligned}$$

have been made. The closed loop poles are the roots of the corresponding modified version of the characteristic polynomial of the system, which is the polynomial in derivatives on the left in (5.9). Analogous remarks hold if we instead consider the dynamics in (5.13), (5.14).

It is easy to see that the resulting new damping and undamped natural frequency in (5.11) and (5.12) can be assigned arbitrary values by varying k_1, k_2. Since such a state feedback controller can always be assumed to be present (and the airframe to have sufficient control surface area and actuator power within the operating envelope of the vehicle) we can assume that (5.13), (5.14), *in fact describes the closed loop system for the pitch channel with the desired (w.r.t. handling qualities) dynamics (i.e. the values of the undamped natural frequency and damping having the desired values).* The values of the quantities in (5.13), (5.14) then no longer represent the physical "bare airframe" properties, but "virtual quantities," synthesized by the controller, cf. below.

Assumption 13.

5.4 Load factor

The simplified linearized pitch plane dynamics model developed so far will be the basis for the pitch channel dynamics in the overall model in Section 2.1. However, the linearized pitch plane dynamics model employs the angle of attack as basic variable and this is in general not a good choice for command, at least not for a piloted aircraft. Therefore, we shall here develop an equivalent model of the pitch channel dynamics which utilize a more natural variable for command, the load factor.

5.4.1 Aerodynamic load factor

As remarked above, in studies of maneuvering aircraft it is often warranted to start by studying the dynamics of the aircraft in a gravity free setting. Since the effects of thrust are easily added later it is moreover convenient to study the dynamics resulting from aerodynamic forces and moments alone. Particularly important is then the load induced by the aerodynamic force in the $\boldsymbol{\sigma}$-direction in B and this can be represented by the *aerodynamic load factor* $\eta^{(a)}$ defined as

$$\eta^{(a)} = \frac{f_L}{mg},$$

where f_L is the lift force in (3.17). Thus, the aerodynamic load factor is the ratio of the lift and weight forces.

We know from earlier remarks that the lift force aerodynamic coefficient C_L in general is essentially linear in α around $\alpha = 0$ and therefore $\eta^{(a)}$ can be described in terms of (the derivative of) C_L (cf. Sec. 5.1.2) as

$$\eta^{(a)} = \frac{1}{2mg}\rho V^2 S_{ref}\frac{\partial C_L}{\partial \alpha}|_{\alpha=0}\alpha \tag{5.15}$$

Moreover, from Section 5.1.2 we know also that for $\alpha = 0$ we may (as a good approximation) replace $\partial C_L/\partial\alpha$ with $-\partial C_Z/\partial\alpha$. Thus, in the language of the linearized model (5.1) (with $\alpha_0 = q_0 = 0$) $\eta^{(a)}$ can be expressed as[4]

$$\eta^{(a)} = -\frac{Z_\alpha}{g}\alpha = -\frac{1}{2mg}\rho V^2 S_{ref}\frac{\partial C_Z}{\partial \alpha}|_{\alpha=0}\alpha \tag{5.16}$$

where $Z_\alpha = Z_\alpha(0)$ and we have used (5.7).

5.4.1.1 State space representation

From (5.13), (5.14) we immediately obtain (after a rescaling of the control variable) a state space model for the behavior of $\eta^{(a)}$ as

$$\frac{d}{dt}\begin{bmatrix} \eta^{(a)} \\ \dot{\eta}^{(a)} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_{sp}^2 & -2\zeta_{sp}\omega_{sp} \end{bmatrix}\begin{bmatrix} \eta^{(a)} \\ \dot{\eta}^{(a)} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_{sp}^2 \end{bmatrix} u. \tag{5.17}$$

By the discussion in (5.3) we know that the expression (5.17) not only describes the open loop behavior but also the closed loop behavior, depending on how we interpret the values of the undamped natural frequency and damping factor ω_{sp}, ζ_{sp}, respectively, and the control variable u. In the open loop case we interpret these as the bare airframe values and in the closed loop case as the values prescribed by the handling qualities, and the control variable u can be interpreted as commanded (desired) aerodynamic load factor $\eta_c^{(a)}$.

From (5.5), (5.7), (5.15), (5.16) and the fact that $Z(0) = 0$ by assumption we also see that the dynamics for the aerodynamic load factor $\eta^{(a)}$ are equivalent (under the stated conditions) to those of the aerodynamic lift force coefficient C_L (or, after a sign change, C_Z). We can therefore by a simple linear transformation $(\alpha, \dot{\alpha}) \to (C_L, \dot{C}_L)$ in (5.17) and obtain a system of exactly the same form and we may interpret the input, after rescaling, as commanded C_L.

The aerodynamic load factor $\eta^{(a)}$ is intimately related to angle of attack α and therefore links the two to "ideal," i.e. gravity free, turning radius. However, for actually representing the motion in E, where gravity is present, it is often more relevant to use the total load factor.

[4]The quantity $\eta^{(a)}/\alpha$, frequently denoted (n/α) is central to handling quality specifications such as CAP and C*, see e.g. (Tobie *et al.*, 1966; MIL-HDBK-1797) and the discussion in (Field, 1993).

5.4.2 Load factor (total)

In piloted flight, the most relevant variable describing the pitch dynamics in most flight conditions is the (total) load factor (Tobie *et al.*, 1966). The *load factor* η represents the total load in the $\boldsymbol{\sigma}$-direction in B and is defined by

$$\eta = \frac{\boldsymbol{\sigma}^T \boldsymbol{f}}{mg} = \frac{f_x \sin(\alpha) - f_z \cos(\alpha)}{mg}. \tag{5.18}$$

Using the simplified equations (4.5)–(4.7) resulting from the assumption $\beta = 0$ we see that (5.18) can be written

$$\eta = -\frac{F_\alpha}{g} = \eta^{(a)} - \frac{F_\alpha^{(gt)}}{g} \tag{5.19}$$

where $F_\alpha^{(gt)}$ is given by (4.8). This shows that controlling η can equally well be viewed as a tracking problem for $\eta^{(a)}$.

5.4.2.1 State space representation

A state space representation of the dynamics for η is more or less immediately obtained from (5.19) and the pitch dynamics model we have developed above (if we use the approximation $\partial C_L/\partial\alpha = -\partial C_Z/\partial\alpha$ near $\alpha = 0$). From (5.17) and (5.19) we obtain the following representation for the dynamics of η,

$$\frac{d}{dt}\begin{bmatrix} \eta \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_{sp}^2 & -2\zeta_{sp}\omega_{sp} \end{bmatrix} \begin{bmatrix} \eta \\ \dot{\eta} \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_{sp}^2 \end{bmatrix} u + \begin{bmatrix} 0 \\ -d \end{bmatrix}, \tag{5.20}$$

where

$$d = \frac{1}{g}\big(\ddot{F}_\alpha^{(gt)} + 2\zeta_{sp}\omega_{sp}\dot{F}_\alpha^{(gt)} + \omega_{sp}^2 F_\alpha^{(gt)}\big). \tag{5.21}$$

We know from the discussion following (5.17) that the dynamics for η will have the form (5.20) in both open and closed loop, and in the latter case the control variable u in (5.20) can be interpreted as commanded aerodynamic load factor $\eta_c^{(a)}$. On a fighter aircraft the terms making up d in (5.21) can be assumed to be directly measurable and therefore can be canceled by the control. Hence, if we define a new control signal $\tilde{u}$ as

$$\tilde{u} = u - \frac{d}{\omega_{sp}} \tag{5.22}$$

the dynamics in (5.20) will be of exactly the same form as in (5.17) and the control variable $\tilde{u}$ can be interpreted as commanded (total) load factor η_c. In other words, the dynamics for $(\alpha, \dot{\alpha})$ (in (5.13)) and $(\eta^{(a)}, \dot{\eta}^{(a)})$ (in (5.17)) and $(\eta, \dot{\eta})$ (in (5.20)) in open loop as well as closed loop are all of *the same form*. Therefore, if we for example interpret the state variables in this system as $(\eta, \dot{\eta})$ in closed loop then the control variable will represent the commanded value η_c for the total load factor (closed loop). One slight problem with this is that we then obtain a differential algebraic system for η, α since α has to be solved for in (5.19) to obtain d in (5.21). However, in practice this can be handled by, at each time step, use values of α obtained from previous time steps to calculate d, then update η using (5.20), then calculate a new value for $\eta^{(a)}$ using (5.19) and from this finally obtain a new value for α from (5.15).

5.4.2.2 Pitch rate

From (4.21) we obtain values for p, r but q is also needed to determine the full vector $\boldsymbol{\omega}$ of angular velocities in B. If we use the simplified relation (4.4) obtained for $\beta = 0$, take time derivative of the sides in the first equality in (5.15) (assuming that ρ, V are slowly time varying) and apply (5.19) we obtain

$$q = -\frac{g}{Z_\alpha}\dot{\eta}^{(a)} + \frac{g}{V}\big(\eta^{(a)} - \frac{F_\alpha^{(gt)}}{g}\big) = -\frac{g}{Z_\alpha}\big(\dot{\eta} + \frac{\dot{F}_\alpha^{(gt)}}{g}\big) + \frac{g}{V}\eta. \tag{5.23}$$

This shows that q can be obtained as an affine output function[5] of the state in either (5.17) or (5.20).

In piloted flight at low Mach numbers, such as take-off and landing, the variable mainly sensed by the pilot is the pitch rate q and therefore it is desirable to be able to control this as well as the load factor. Indeed, the handling quality criteria based on the C* parameter (Tobie *et al.*, 1966; Field, 1993) are based on this insight. However, using (5.23) and (5.17) or (5.20) it is straightforward to design controllers (and thereby models of the closed loop behavior in the pitch channel) based on the pitch rate, or a mix of the pitch rate and load factor.

5.4.3 Computation of the term $\dot{F}_\alpha^{(gt)}$

Assumption 14.

If we assume that α and $f_z^{(t)}$ are slowly varying[6] compared to the variation of $f_x^{(g)}, f_z^{(g)}$ we obtain, after taking time derivatives of both sides of (4.8), that

$$\dot{F}_\alpha^{(gt)} = \frac{1}{m}\big(\dot{f}_z^{(g)}\cos(\alpha) - \dot{f}_x^{(g)}\sin(\alpha)\big). \tag{5.24}$$

To obtain expressions for $\dot{f}_x^{(g)}, \dot{f}_z^{(g)}$ we note that the forces $\boldsymbol{f}$ in B and their counterparts $\boldsymbol{F}$ in E are related by the same transformation rule as the velocities in (3.4), viz.

$$\boldsymbol{F} = \boldsymbol{R}(\boldsymbol{q})\boldsymbol{f}$$

and it follows that (cf. Appendix A)

$$\dot{\boldsymbol{F}} = \boldsymbol{R}(\boldsymbol{q})(\boldsymbol{\omega} \times \boldsymbol{f}) + \boldsymbol{R}(\boldsymbol{q})\dot{\boldsymbol{f}}. \tag{5.25}$$

We can decompose $\boldsymbol{F}$ into aerodynamic components $\boldsymbol{F}^{(a)}$, gravity $\boldsymbol{F}^{(g)}$ and thrust $\boldsymbol{F}^{(t)}$ as

$$\boldsymbol{F} = \boldsymbol{F}^{(a)} + \boldsymbol{F}^{(g)} + \boldsymbol{F}^{(t)}$$

and applying this to (5.25) (assuming gravity in E is constant so that $\dot{\boldsymbol{F}}^{(g)} = \boldsymbol{0}$) we obtain in particular

$$\dot{\boldsymbol{f}}^{(g)} = -\boldsymbol{\omega} \times \boldsymbol{f}^{(g)}. \tag{5.26}$$

Thus, a simplified model for $\dot{F}_\alpha^{(gt)}$ is obtained from (5.24) and (5.26).

5.5 Velocity

By combining (4.9)–(4.12) we obtain

$$F_V = \frac{1}{m}\big(+ (f_x^{(t)} + f_x^{(g)})\cos(\alpha) + f_z^{(g)}\sin(\alpha) - f_D\big). \tag{5.27}$$

[5] The zero of the corresponding transfer function is known to have a significant effect on handling qualities and the associated time constant, often denoted T_{θ_2} (MIL-HDBK-1797), is in body frame quantities represented by $-V/Z_\alpha$.

[6] In other words, we assume that the part of $\dot{F}_\alpha^{(gt)}$ which is due to $\dot{\alpha}, \dot{f}_z^{(t)}$ is small compared to the rest of $\dot{F}_\alpha^{(gt)}$. This is a reasonable approximation in those instances when $\dot{F}_\alpha^{(gt)}$ yields a noticeable contribution to (5.19) and $\dot{\eta}$ is not small. (The assumption about slowly varying α and $f_z^{(t)}$ is used only here.)

This is the expression for F_V used in the model in Section 2.1.

The $f_x^{(t)}$-term in (5.27) can be modeled as[7]

$$\dot{f}_x^{(t)} = \frac{1}{\tau_T}(t_{ss} f_{T_0}(M,h) - f_x^{(t)}), \tag{5.28}$$

where $t_{ss} \in [0,1]$ is the throttle stick setting, $f_{T_0}(M,h)$ is the maximum engine thrust at Mach number M and altitude h, and τ_T is the engine response time constant.

The drag force f_D in (3.18) and (5.27) is modeled in terms of the drag force coefficient C_D as

$$f_D = \frac{1}{2}\rho S_{ref} C_D(\alpha, M) \tag{5.29}$$

where

$$C_D(\alpha, M) = C_{D_0}(M) + K(M) C_L(\alpha)^2 \tag{5.30}$$

and the parasitic drag coefficient $C_{D_0}(M)$ and the induced drag quadratic factor $K(M)$ both have some prescribed dependencies on Mach number (see e.g. (Raymer, 2006, Sec. 12.5, 12.6)). The α-dependence in C_L can be expressed in terms of $\eta^{(a)}$ using (5.15) (with $C_Z(0) = 0$, cf. Sec. 5.1.2).

5.6 Load Limiter

On a piloted aircraft, the flight envelope is determined by constraints on the pilot, airframe and engine. The engine induced limitations are easy to incorporate into the engine thrust model (5.28). The constraints induced by pilot and airframe can to a large extent be captured by limits on the load factor alone, due to the intimate relation between angle of attack, aerodynamic load factor and load factor, cf. (5.15) and (5.19).

5.6.1 Predictive limiting

Since there is a dynamic relation between load factor command η_c and the load factor η it is nontrivial to determine which values of the command η_c that will yields future values of η which are all admissible, i.e. satisfy all the constraints. One way to address this problem is to solve the differential equation which describes the evolution of η for given initial values and varying inputs η_c, and seek inputs that give extrema in the future values of η. If the extrema in η are non admissible, the input η_c needs to be constrained, and the process can be repeated until a sufficiently restrictive constraint on the input is found such that the future output is admissible at all times.

Fortunately, the dynamics for η_c and η are linear (time invariant) second order, so finding the extrema η and corresponding "worst-case" inputs η_c is not difficult to do, either analytically or numerically. In particular, max and min values of η, when η_c is constrained to an interval, can be generated with constant inputs.

Furthermore, in the case where the load factor dynamics are "ideal" the damping ζ_{sp} is in the order of 0.7 and then there is very little "overshoot" in the step response, cf. Figure 5.1 below, and load factor limiting essentially boils down to limiting the command η_c. (This holds true even for fairly large variations in the initial derivative $\dot{\eta}(t_0)$, as can be seen in Figure 5.1.)

However, the set of admissible load factor values varies significantly throughout the envelope so in effect the check for admissibility of the input should be done very often, ideally at each time step in the solver used when executing the model in Section 2.1.

[7] An extra parameter, signifying whether or not the afterburner is lit, is often added.

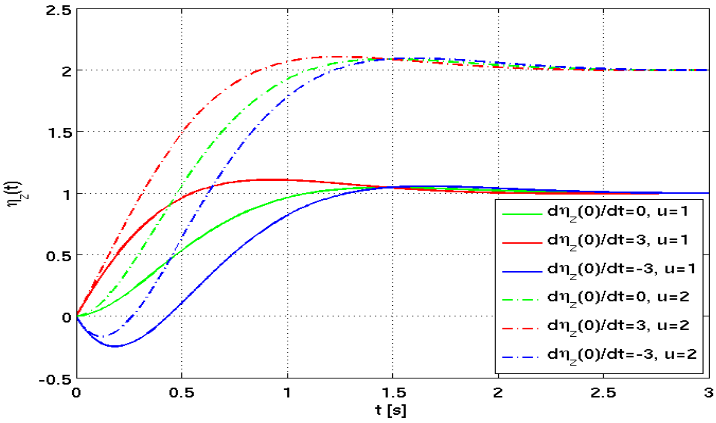

Figure 5.1: Typical step response for $\eta(t)$ as in (5.20), (5.22) with parameters $\omega_{sp} = 3, \zeta_{sp} = 1/\sqrt{2}$ and $\eta(t_0) = 0$, and $\tilde{u} = 1, 2$.

6 Autopilot

The model developed in the previous chapters can be taken to describe the closed loop behavior of an aircraft with stability augmenting, or even dynamics synthesizing, flight control system. However, this assumes that the commanded input is given in terms of values for the (velocity axis) roll rate $p^{(W)}$, load factor η and throttle setting t_{ss}. These values can be given by a pilot, or operator, in a manned simulation but it is also occasionally useful to have an autopilot to generate acceleration commands which are to be followed by the aircraft. In this chapter we shall develop an autopilot for the simplest case of path following, namely that of adjusting the velocity vector of the aircraft in E until it coincides with a reference velocity vector. The most complex part of the autopilot design is an orientation command generator which mimics the actions of a pilot in order to generate commands for $p^{(W)}$, η and t_{ss} that can be directly fed to the aircraft model.

6.1 Velocity direction following

We assume that the object is to have the aircraft velocity $\boldsymbol{V}$ in E approach a reference value $\boldsymbol{V}_r$, which may be constant or slowly time varying. This problem is solved in two steps. The first step is a guidance law that provides acceleration commands which when followed will align the aircraft velocity vector with a reference velocity vector (without regards to the magnitude). The second step is a command generator that provides commands for roll rate, load factor and throttle setting to perform this alignment and adjust the magnitude of the aircraft velocity vector to the reference value. The latter subproblem is solved separately so for the most part in the development below we shall assume that the magnitude of the aircraft velocity is constant (or slowly varying).

6.1.1 Guidance law

Let the *heading error angle* $\theta \in [-\pi, \pi)$ be defined by

$$\cos(\theta) = \frac{\boldsymbol{V}_r^T \boldsymbol{V}}{\|\boldsymbol{V}_r\| \|\boldsymbol{V}\|} \tag{6.1}$$

where it is assumed that $\boldsymbol{V}, \boldsymbol{V}_r$ are bounded away from $\boldsymbol{0}$. The first step in developing the autopilot is to devise a smooth guidance law $\boldsymbol{G} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ such that θ will be driven to 0 when $\boldsymbol{G}$ defines the dynamics for $\boldsymbol{V}$, i.e. when the aircraft acceleration $\dot{\boldsymbol{V}}$ is given by

$$\dot{\boldsymbol{V}} = \boldsymbol{G}(\boldsymbol{V}, \boldsymbol{V}_r). \tag{6.2}$$

Define the vector $\boldsymbol{\Omega}_g$ in E by

$$\boldsymbol{\Omega}_g = \boldsymbol{V} \times \boldsymbol{V}_r$$

and consider the guidance law $\boldsymbol{G}$ given by

$$\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{V}_r) = \frac{c_g}{\|\boldsymbol{V}\|^2 \|\boldsymbol{V}_r\|} (\boldsymbol{\Omega}_g \times \boldsymbol{V}), \tag{6.3}$$

where $c_g > 0$ is a constant. We then have[1]

$$\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{V}_r) = \frac{c_g}{\|\boldsymbol{V}\|^2\|\boldsymbol{V}_r\|}\big(\boldsymbol{V} \times (\boldsymbol{V}_r \times \boldsymbol{V})\big) = \frac{c_g}{\|\boldsymbol{V}_r\|}\boldsymbol{P}_{[\boldsymbol{V}]^\perp}\boldsymbol{V}_r = \frac{c_g}{\|\boldsymbol{V}_r\|}(\boldsymbol{I} - \boldsymbol{P}_{[\boldsymbol{V}]})\boldsymbol{V}_r. \quad (6.4)$$

It follows that the magnitude of $\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{V}_r)$ is given by

$$\|\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{V}_r)\|^2 = \frac{c_g^2}{\|\boldsymbol{V}_r\|^2}\Big\|\boldsymbol{V}_r - \frac{\boldsymbol{V}^T\boldsymbol{V}_r}{\|\boldsymbol{V}\|^2}\boldsymbol{V}\Big\|^2 = \frac{c_g^2}{\|\boldsymbol{V}_r\|^2}(\|\boldsymbol{V}_r\|^2 - 2\frac{(\boldsymbol{V}^T\boldsymbol{V}_r)^2}{\|\boldsymbol{V}\|^2} + \frac{(\boldsymbol{V}^T\boldsymbol{V}_r)^2}{\|\boldsymbol{V}\|^2}) = c_g^2(1 - \cos^2(\theta)) \quad (6.5)$$

and we see that $\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{V}_r) = \boldsymbol{0}$ is equivalent to $\cos(\theta) = \pm 1$. (There are thus two equilibrium points to (6.2) which can be defined in terms on this condition on θ.) Further, since the right hand side in (6.5) is bounded by c_g^2 for $\cos(\theta) \geq 0$ it is clear that the constant c_g determines the maximal magnitude of the acceleration commands. When the guidance law $\boldsymbol{G}$ in (6.3) is applied so that (6.2) holds we see that

$$\dot{\boldsymbol{V}} = \frac{c_g}{\|\boldsymbol{V}_r\|}\boldsymbol{P}_{[\boldsymbol{V}]^\perp}\boldsymbol{V}_r \quad (6.6)$$

and it follows in particular that

$$\frac{d}{dt}\frac{\|\boldsymbol{V}\|^2}{2} = \boldsymbol{V}^T\dot{\boldsymbol{V}} = 0. \quad (6.7)$$

so the guidance law does not require the velocity of the aircraft to change, i.e. $\boldsymbol{G}$ is consistent with *normal acceleration maneuvering*.

6.1.1.1 Stability of the guidance law

To see that the guidance law $\boldsymbol{G}$ in (6.3) actually accomplishes the goal of driving the heading error angle θ in (6.1) to 0 in the simplest case we assume that $\dot{\boldsymbol{V}}_r = \boldsymbol{0}$. The dynamics for $\cos(\theta) \in [-1, 1]$ when $\boldsymbol{G}$ is applied as in (6.2) are then (since $\dot{V} = d\|\boldsymbol{V}\|/dt = 0$ by (6.7))

$$\frac{d}{dt}\cos(\theta) = \frac{\boldsymbol{V}_r^T\dot{\boldsymbol{V}}}{\|\boldsymbol{V}_r\|\|\boldsymbol{V}\|} = \frac{c_g}{\|\boldsymbol{V}_r\|^2\|\boldsymbol{V}\|}\boldsymbol{V}_r^T\boldsymbol{P}_{[\boldsymbol{V}]^\perp}\boldsymbol{V}_r = \frac{c_g}{\|\boldsymbol{V}_r\|^2\|\boldsymbol{V}\|}\boldsymbol{V}_r^T(\boldsymbol{I} - \boldsymbol{P}_{[\boldsymbol{V}]})\boldsymbol{V}_r = \frac{c_g}{\|\boldsymbol{V}\|}\big(1 - (\frac{\boldsymbol{V}_r^T\boldsymbol{V}}{\|\boldsymbol{V}_r\|\|\boldsymbol{V}\|})^2\big) = \frac{c_g}{V}(1 - \cos^2(\theta)) = \frac{c_g}{V}(1 - \cos(\theta))(1 + \cos(\theta)). \quad (6.8)$$

This differential equation has two equilibria given by $\cos(\theta) = \pm 1$ and for $\cos(\theta) \in (-1, 1)$ we have $d\cos(\theta)/dt > 0$. Hence, the equilibrium at $\cos(\theta) = 1$ is stable with domain of attraction $(-1, 1]$ (and consequently the equilibrium at $\cos(\theta) = -1$ is unstable).

[1] This shows that $\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{V}_r)$ is independent of the magnitudes of $\boldsymbol{V}, \boldsymbol{V}_r$ which is the reason for the normalization chosen in the definition (6.3).

Linearized dynamics

A Taylor expansion of both sides of (6.8) around $\theta = 0$ shows that we have the linear approximation

$$\dot{\theta} = -\frac{c_g}{2V}\theta. \tag{6.9}$$

It follows that the quantity $2V/c_g$ can be interpreted as a time constant [2] for the dynamics of θ near 0.

Time varying reference velocity

In case $\boldsymbol{V}_r$ is not constant we define[3] the angular velocity $\boldsymbol{\Omega}_r$ in E by

$$\frac{d}{dt}\frac{\boldsymbol{V}_r}{\|\boldsymbol{V}_r\|} = \boldsymbol{\Omega}_r \times \frac{\boldsymbol{V}_r}{\|\boldsymbol{V}_r\|}. \tag{6.10}$$

and note that the dynamics for $\cos(\theta)$ then become (cf. (6.8))

$$\frac{d}{dt}\cos(\theta) = \frac{d}{dt}\left(\frac{\boldsymbol{V}_r^T}{\|\boldsymbol{V}_r\|}\right)\frac{\boldsymbol{V}}{\|\boldsymbol{V}\|} + \frac{\boldsymbol{V}_r^T\dot{\boldsymbol{V}}}{\|\boldsymbol{V}_r\|\|\boldsymbol{V}\|} = \\ (\boldsymbol{\Omega}_r \times \frac{\boldsymbol{V}_r}{\|\boldsymbol{V}_r\|})^T\frac{\boldsymbol{V}}{\|\boldsymbol{V}\|} + \frac{c_g}{V}(1-\cos^2(\theta)). \tag{6.11}$$

The value of $\cos(\theta)$ increases when the right hand side is positive, i.e. when

$$\cos^2(\theta) < 1 + \frac{1}{c_g}(\boldsymbol{\Omega}_r \times \frac{\boldsymbol{V}_r}{\|\boldsymbol{V}_r\|})^T\boldsymbol{V}$$

so in particular $\cos(\theta)$ always increases as long as the rightmost term here is positive. In case it is negative we can estimate its size by using the bound

$$|(\boldsymbol{\Omega}_r \times \frac{\boldsymbol{V}_r}{\|\boldsymbol{V}_r\|})^T\boldsymbol{V}| \le \|\boldsymbol{\Omega}_r \times \frac{\boldsymbol{V}_r}{\|\boldsymbol{V}_r\|}\|\|\boldsymbol{V}\| \le \|\boldsymbol{\Omega}_r\|\|\boldsymbol{V}\|.$$

Thus, if $\|\boldsymbol{\Omega}_r\|$ is bounded we can make $\cos(\theta)$ arbitrarily close to 1 (i.e. make the heading error arbitrarily close to 0) by making c_g large.

6.1.2 Orientation command generator

The second step in the development of an autopilot is to devise a command generator for the roll and pitch commands needed in order to follow the acceleration commands provided by the guidance law.

Let $\dot{\boldsymbol{v}}_g$ be the value of the guidance law in (6.3) after rotation into B, i.e.

$$\dot{\boldsymbol{v}}_g = \boldsymbol{R}(\boldsymbol{q})^T\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{V}_r) \tag{6.12}$$

where $\boldsymbol{R}(\boldsymbol{q})$ is the rotation matrix in (3.4), and let $\boldsymbol{v}_r$ be the reference velocity in B, i.e.

$$\boldsymbol{v}_r = \boldsymbol{R}(\boldsymbol{q})^T\boldsymbol{V}_r. \tag{6.13}$$

[2] For motion with only normal acceleration, the magnitude of the acceleration is bilinear in the magnitudes of the velocity and angular velocity. Thus, for constant normal acceleration the velocity and angular velocity are (modulo a constant) reciprocal. Since $\dot{\theta}$ plays the role of angular velocity here the constant acceleration condition suggests that, for given value of θ, the time derivative $\dot{\theta}$ should (modulo a constant) be reciprocal to V, as in (6.9).

[3] Let $\boldsymbol{\nu}$ be a smoothly time varying unit norm vector in $\mathbb{R}^3$. Then $\dot{\boldsymbol{\nu}}^T\boldsymbol{\nu} = 0$ at all times. Define $\boldsymbol{\xi}$ by $\boldsymbol{\xi} = \boldsymbol{\nu} \times \dot{\boldsymbol{\nu}}$. Then $\boldsymbol{\xi} \times \boldsymbol{\nu} = (\boldsymbol{\nu} \times \dot{\boldsymbol{\nu}}) \times \boldsymbol{\nu} = \boldsymbol{\nu} \times (\dot{\boldsymbol{\nu}} \times \boldsymbol{\nu}) = \dot{\boldsymbol{\nu}}$.

Then we have

$$\dot{\boldsymbol{v}}_g = \frac{c_g}{\|\boldsymbol{V}\|^2\|\boldsymbol{V}_r\|}\boldsymbol{R}(\boldsymbol{q})^T\big(\boldsymbol{V}\times(\boldsymbol{V}_r\times\boldsymbol{V})\big) = \frac{c_g}{\|\boldsymbol{v}\|^2\|\boldsymbol{v}_r\|}\big(\boldsymbol{v}\times(\boldsymbol{v}_r\times\boldsymbol{v})\big) = \frac{c_g}{\|\boldsymbol{v}_r\|}\boldsymbol{P}_{[\boldsymbol{v}]^\perp}\boldsymbol{v}_r \quad (6.14)$$

where we have used the well-known fact that rotations commute with the cross product operation (cf. e.g. (Arnold, 1989, p. 131)). The relations (6.14) show in particular that $\dot{\boldsymbol{v}}_g \in [\boldsymbol{v}]^\perp$ at all times, as expected, and to drive $\boldsymbol{G}(\boldsymbol{V},\boldsymbol{V}_r)$ to $\mathbf{0}$ in E is equivalent to drive $\dot{\boldsymbol{v}}_g$ to $\mathbf{0}$ in B.

Principle

The principle used to drive $\dot{\boldsymbol{v}}_g$ to $\mathbf{0}$ is easy to illustrate if we note that when guidance commands are exactly followed then the identity (6.2) holds. From (3.5), (3.6) and (6.12) we see that we must then have

$$\dot{\boldsymbol{v}}_g = \frac{1}{m}\boldsymbol{f}. \quad (6.15)$$

However, since $\beta \equiv 0$ we have $\boldsymbol{e}_{\boldsymbol{\lambda}} = \boldsymbol{e}_2$ and therefore the left hand side lies in $[\boldsymbol{e}_2,\boldsymbol{\sigma}]$ at all times. The right hand side of (6.15) can be expressed in wind axes components as

$$\frac{1}{m}\boldsymbol{f} = \frac{1}{m}\boldsymbol{e}_{\boldsymbol{v}}^T\boldsymbol{f}\boldsymbol{e}_{\boldsymbol{v}} + \frac{1}{m}\boldsymbol{e}_2^T\boldsymbol{f}\boldsymbol{e}_2 + \frac{1}{m}\boldsymbol{\sigma}^T\boldsymbol{f}\boldsymbol{\sigma}$$
$$= \frac{1}{m}\boldsymbol{e}_{\boldsymbol{v}}^T\boldsymbol{f}\boldsymbol{e}_{\boldsymbol{v}} + \frac{f_y^{(g)}}{m}\boldsymbol{e}_2 + g\eta\boldsymbol{\sigma} = \frac{1}{m}\boldsymbol{e}_{\boldsymbol{v}}^T\boldsymbol{f}\boldsymbol{e}_{\boldsymbol{v}} - \|\boldsymbol{v}\|r^{(W)}\boldsymbol{e}_2 + g\eta\boldsymbol{\sigma} \quad (6.16)$$

where we have used the fact that $f_y^{(a)} = 0$ when $\beta = 0$, that thrust does not act in the body y-axis direction (cf. Sec. 4.1.1), relation (4.3) as well as the the definition of load factor (5.18). Now, the first term on the right in (6.16) can be made zero by proper adjustment of thrust (to balance drag) and the remaining two terms must then sum up to $\dot{\boldsymbol{v}}_g$ in order to have equality in (6.15). The last term on the right (6.16) is directly controllable (by applying the proper commands in pitch) but the middle term in (6.16) can only be affected indirectly, by rolling the aircraft (so that the gravity force component in the body y-axis takes the appropriate value). In essence, the aircraft must be rolled so that the body force component in the y-direction takes the proper value and the appropriate pitch command must be applied to satisfy the force requirement in the $\boldsymbol{\sigma}$-direction. Next we shall devise a controller for this.

6.1.2.1 Command generator control law

From (6.12) and the theory of relative motion (Appendix A) it follows that

$$\ddot{\boldsymbol{v}}_g = \boldsymbol{R}(\boldsymbol{q})^T\frac{d}{dt}\boldsymbol{G}(\boldsymbol{V},\boldsymbol{V}_r) - \boldsymbol{\omega}\times\dot{\boldsymbol{v}}_g. \quad (6.17)$$

The time derivative on the right can be computed explicitly from (6.4) as

$$\frac{d}{dt}\boldsymbol{G}(\boldsymbol{V},\boldsymbol{V}_r) = \frac{c_g}{\|\boldsymbol{V}\|^2\|\boldsymbol{V}_r\|}\Big(-\big(\frac{2}{\|\boldsymbol{V}\|}\frac{d}{dt}\|\boldsymbol{V}\| + \frac{1}{\|\boldsymbol{V}_r\|}\frac{d}{dt}\|\boldsymbol{V}_r\|\big)\big(\boldsymbol{V}\times(\boldsymbol{V}_r\times\boldsymbol{V})\big)$$
$$+\big(\big(\dot{\boldsymbol{V}}\times(\boldsymbol{V}_r\times\boldsymbol{V})\big) + \big(\boldsymbol{V}\times(\dot{\boldsymbol{V}}_r\times\boldsymbol{V})\big) + \big(\boldsymbol{V}\times(\boldsymbol{V}_r\times\dot{\boldsymbol{V}})\big)\big)\Big). \quad (6.18)$$

In the important special case of constant reference velocity and normal acceleration maneuvering, i.e.

$$\dot{\boldsymbol{V}}_r = \boldsymbol{0}, \quad \dot{V} = \frac{d\|\boldsymbol{V}\|}{dt} = 0, \tag{6.19}$$

the right hand side of (6.18) simplifies considerably. We shall now in a sequence of steps expand and simplify the right hand side of (6.17) using the representation (6.18), under the assumption (6.19).

Normal acceleration and constant reference velocity

When (6.19) holds we have $0 = \boldsymbol{V}^T\dot{\boldsymbol{V}} = \boldsymbol{v}^T\boldsymbol{R}(\boldsymbol{q})^T\dot{\boldsymbol{V}}$ and thus

$$\begin{aligned}
\boldsymbol{R}(\boldsymbol{q})^T \frac{d}{dt}\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{V}_r) &= \\
&\frac{c_g}{\|\boldsymbol{v}\|^2\|\boldsymbol{v}_r\|}\Big(\big((\boldsymbol{R}(\boldsymbol{q})^T\dot{\boldsymbol{V}} \times (\boldsymbol{v}_r \times \boldsymbol{v})\big) + \big(\boldsymbol{v} \times (\boldsymbol{v}_r \times \boldsymbol{R}(\boldsymbol{q})^T\dot{\boldsymbol{V}})\big)\Big) \\
&= -\frac{c_g}{\|\boldsymbol{v}\|^2\|\boldsymbol{v}_r\|}\big((\boldsymbol{v}^T\boldsymbol{v}_r)\boldsymbol{R}(\boldsymbol{q})^T\dot{\boldsymbol{V}} + (\boldsymbol{v}_r^T\boldsymbol{R}(\boldsymbol{q})^T\dot{\boldsymbol{V}})\boldsymbol{v}\big),
\end{aligned} \tag{6.20}$$

where we have used (6.13), the fact that cross products commute with rotations and the triple product formula. Further, from the theory of relative motion we have the relation (3.7) and thus (6.20) can be written

$$\begin{aligned}
\boldsymbol{R}(\boldsymbol{q})^T \frac{d}{dt}\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{V}_r) = &-\frac{c_g}{\|\boldsymbol{v}\|^2\|\boldsymbol{v}_r\|}\big((\boldsymbol{v}^T\boldsymbol{v}_r)\dot{\boldsymbol{v}} + (\dot{\boldsymbol{v}}^T\boldsymbol{v}_r)\boldsymbol{v}\big) \\
&- \frac{c_g}{\|\boldsymbol{v}\|^2\|\boldsymbol{v}_r\|}\big((\boldsymbol{v}^T\boldsymbol{v}_r)(\boldsymbol{\omega} \times \boldsymbol{v}) + ((\boldsymbol{\omega} \times \boldsymbol{v})^T\boldsymbol{v}_r)\boldsymbol{v}\big).
\end{aligned} \tag{6.21}$$

Coordinated turn

The term $\boldsymbol{\omega} \times \boldsymbol{v}$ in (6.21) can be expressed in terms of wind axes quantities as

$$\boldsymbol{\omega} \times \boldsymbol{v} = (p^{(W)}\boldsymbol{e}_{\boldsymbol{v}} + q\boldsymbol{e}_2 + r^{(W)}\boldsymbol{\sigma}) \times \boldsymbol{v} = q\|\boldsymbol{v}\|\boldsymbol{\sigma} - r^{(W)}\|\boldsymbol{v}\|\boldsymbol{e}_2 \tag{6.22}$$

(since $\boldsymbol{e}_{\boldsymbol{v}}, \boldsymbol{e}_2, \boldsymbol{\sigma}$ is a left handed system, $\boldsymbol{e}_{\boldsymbol{v}} \times \boldsymbol{e}_2 = -\boldsymbol{\sigma}$). Due to the condition $\beta = 0$ the last term on the right in (6.22) is directly related to $f_y^{(g)}$ as in (4.3), indeed

$$\frac{f_y^{(g)}}{m} = -\boldsymbol{\omega}^T\boldsymbol{\sigma}\|\boldsymbol{v}\| = -r^{(W)}\|\boldsymbol{v}\|. \tag{6.23}$$

The relation (6.23) can also be expressed in terms of Euler angles. Let $\Psi, \Phi \in [-\pi, \pi)$ and $\Theta \in [-\pi/2, \pi/2)$ be the yaw, roll and pitch angles, respectively, in the yaw-pitch-roll decomposition of $\boldsymbol{R}(\boldsymbol{q})^T$ given by[4]

$$\boldsymbol{R}(\boldsymbol{q})^T = \boldsymbol{R}_\Phi\boldsymbol{R}_\Theta\boldsymbol{R}_\Psi$$

where the rotation matrices $\boldsymbol{R}_\Psi, \boldsymbol{R}_\Theta, \boldsymbol{R}_\Phi$ represent the elementary yaw, pitch and roll axis rotation matrices, respectively (Stevens & Lewis, 2003, pp. 26,27). Then we have

$$\begin{aligned}
&\boldsymbol{R}_\Phi\boldsymbol{R}_\Theta\boldsymbol{R}_\Psi = \\
&\begin{bmatrix}
c(\Theta)c(\Psi) & c(\Theta)s(\Psi) & -s(\Theta) \\
-c(\Phi)s(\Psi) + s(\Phi)s(\Theta)c(\Psi) & c(\Phi)c(\Psi) + s(\Phi)s(\Theta)s(\Psi) & s(\Phi)c(\Theta) \\
s(\Phi)s(\Psi) + c(\Phi)s(\Theta)c(\Psi) & -s(\Phi)c(\Psi) + c(\Phi)s(\Theta)s(\Psi) & c(\Phi)c(\Theta)
\end{bmatrix}
\end{aligned} \tag{6.24}$$

[4]Thus assumes the standard flight mechanical convention of having the z-axis in E pointing downwards.

where we have let $c(\cdot)$ and $s(\cdot)$ denote $\cos(\cdot)$ and $\sin(\cdot)$ respectively. It follows in particular that

$$\boldsymbol{f}^{(g)} = mg[-\sin(\Theta), \sin(\Phi)\cos(\Theta), \cos(\Phi)\cos(\Theta)]^T$$

so that (6.23) gives

$$-r^{(W)}\|\boldsymbol{v}\| = g\sin(\Phi)\cos(\Theta). \tag{6.25}$$

For later reference we note that that the relation between Euler angular rates and body rates which follows from (6.24) is (Stevens & Lewis, 2003)

$$\begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{bmatrix} = \begin{bmatrix} 1 & \tan(\Theta)\sin(\Phi) & \tan(\Theta)\cos(\Phi) \\ 0 & \cos(\Phi) & -\sin(\Phi) \\ 0 & \sin(\Phi)/\cos(\Theta) & \cos(\Phi)/\cos(\Theta) \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}. \tag{6.26}$$

Dynamics for $\dot{\boldsymbol{v}}_g$

We now turn to the last term on the right in (6.17). If we express it in terms of wind axes quantities we have (since $\dot{\boldsymbol{v}}_g \in [\boldsymbol{v}]^\perp$)

$$\begin{aligned} \boldsymbol{\omega} \times \dot{\boldsymbol{v}}_g &= (p^{(W)}\boldsymbol{e}_{\boldsymbol{v}} + q\boldsymbol{e}_2 + r^{(W)}\boldsymbol{\sigma}) \times (\dot{v}_{g,2}\boldsymbol{e}_2 + (\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)\boldsymbol{\sigma}) = \\ & p^{(W)}\boldsymbol{e}_{\boldsymbol{v}} \times (\dot{v}_{g,2}\boldsymbol{e}_2 + (\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)\boldsymbol{\sigma}) + q\boldsymbol{e}_2 \times (\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)\boldsymbol{\sigma} + r^{(W)}\boldsymbol{\sigma} \times \dot{v}_{g,2}\boldsymbol{e}_2 = \\ & (r^{(W)}\dot{v}_{g,2} - q(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g))\boldsymbol{e}_{\boldsymbol{v}} + p^{(W)}(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)\boldsymbol{e}_2 - p^{(W)}\dot{v}_{g,2}\boldsymbol{\sigma} \end{aligned} \tag{6.27}$$

where $\dot{v}_{g,2} = \boldsymbol{e}_2^T\dot{\boldsymbol{v}}_g$. Thus, from (6.17), (6.21), (6.22) and (6.25), (6.27) we obtain

$$\begin{aligned} \ddot{\boldsymbol{v}}_g = -\frac{c_g}{V}\cos(\theta)\dot{\boldsymbol{v}} - c_g\frac{\dot{\boldsymbol{v}}^T\boldsymbol{v}_r}{V\|\boldsymbol{v}_r\|}\boldsymbol{e}_{\boldsymbol{v}} - \frac{c_g}{V}\cos(\theta)(qV\boldsymbol{\sigma} + g\sin(\Phi)\cos(\Theta)\boldsymbol{e}_2) \\ - \frac{c_g}{V\|\boldsymbol{v}_r\|}(qV(\boldsymbol{\sigma}^T\boldsymbol{v}_r) + gv_{r,2}\sin(\Phi)\cos(\Theta))\boldsymbol{e}_{\boldsymbol{v}} \\ + (q(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g) + \frac{g\dot{v}_{g,2}}{V}\sin(\Phi)\cos(\Theta))\boldsymbol{e}_{\boldsymbol{v}} \\ - p^{(W)}(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)\boldsymbol{e}_2 + p^{(W)}\dot{v}_{g,2}\boldsymbol{\sigma}, \end{aligned} \tag{6.28}$$

where $v_{r,2} = \boldsymbol{e}_2^T\boldsymbol{v}_r$. This is the expression for $\ddot{\boldsymbol{v}}_g$ that we are going to use as a basis for our command generator control law.

Separated dynamics for $\dot{\boldsymbol{v}}_g$

It is easy to see that $\dot{\boldsymbol{\sigma}} \in [\boldsymbol{v}]$ when $\beta = 0$ and therefore (recall that $\dot{\boldsymbol{v}}_g \in [\boldsymbol{v}]^\perp$) we obtain from (3.1), (6.22) and (6.28) that

$$\begin{aligned} \frac{d}{dt}(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g) = \dot{\boldsymbol{\sigma}}^T\dot{\boldsymbol{v}}_g + \boldsymbol{\sigma}^T\ddot{\boldsymbol{v}}_g = \\ -\frac{c_g}{V}\cos(\theta)\boldsymbol{\sigma}^T(\frac{1}{m}\boldsymbol{f} - \boldsymbol{\omega} \times \boldsymbol{v}) - c_g q\cos(\theta) + p^{(W)}\dot{v}_{g,2} = \\ -\eta g\frac{c_g}{V}\cos(\theta) + p^{(W)}\dot{v}_{g,2}. \end{aligned} \tag{6.29}$$

From (6.28) we also obtain (since $\dot{\boldsymbol{v}} \in [\boldsymbol{e}_2]^\perp$, cf. Sec. 4.1.1)

$$\ddot{v}_{g,2} = \boldsymbol{e}_2^T\ddot{\boldsymbol{v}}_g = -g\frac{c_g}{V}\cos(\theta)\sin(\Phi)\cos(\Theta) - p^{(W)}\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g. \tag{6.30}$$

and from (6.26) have

$$\frac{d}{dt}(\sin(\Phi)\cos(\Theta)) = \dot{\Phi}\cos(\Phi)\cos(\Theta) - \dot{\Theta}\sin(\Phi)\sin(\Theta) =$$
$$\cos(\Phi)\cos(\Theta)(p + \tan(\Theta)\sin(\Phi)q + \tan(\Theta)\cos(\Phi)r)$$
$$- \sin(\Phi)\sin(\Theta)(\cos(\Phi)q - \sin(\Phi)r) =$$
$$p\cos(\Phi)\cos(\Theta) + r\sin(\Theta). \quad (6.31)$$

The quantity $\cos(\theta)$ defined in (6.1), which occurs in many places above, can be expressed in terms of the quantities $\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g$ and $\dot{v}_{g,2} = \boldsymbol{e}_2^T\dot{\boldsymbol{v}}_g$ if we note that (6.5) and (6.12) give

$$\|\dot{\boldsymbol{v}}_g\|^2 = c_g^2(1 - \cos^2(\theta))$$

and therefore

$$\cos(\theta) = \pm\sqrt{1 - \frac{\|\dot{\boldsymbol{v}}_g\|^2}{c_g^2}} = \pm\sqrt{1 - \frac{(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)^2 + (\dot{v}_{g,2})^2}{c_g^2}}. \quad (6.32)$$

We note that $\cos(\theta)$ does not change sign in the open set

$$\mathcal{C} = \{(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g, \dot{v}_{g,2}) \subset \mathbb{R}^2 \,|\, (\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)^2 + (\dot{v}_{g,2})^2 < c_g^2\}. \quad (6.33)$$

Summing up, the open loop dynamics for the command generator are given by (6.29)–(6.31) with the representation (6.32).

Small Euler angles assumption

We are going to partly linearize the open loop dynamics before specifying the control law. More specifically, we are going to use the linearizations of (6.30) and (6.31) around $\Phi = \Theta = 0$, which are given by

$$\ddot{v}_{g,2} = -g\frac{c_g}{V}\Phi\cos(\theta) - p^{(W)}\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g, \quad (6.34)$$
$$\dot{\Phi} = p. \quad (6.35)$$

The open loop dynamics used by the command generator control law are thus given by (6.29), (6.34) and (6.35) with the representation (6.32). We note here also that as a linear approximation around $\alpha = 0$ we have

$$p^{(W)} = p$$

which is useful when implementing the command generator control law below.

Command generator control law

The command generator control law for roll and pitch employs body roll rate p and normal acceleration η as control variables. It is nonlinear but resembles[5] (apart from a nonlinear scale factor) a linear state feedback law, viz.

$$p = g\frac{c_g}{V}(\gamma_1\dot{v}_{g,2} + \gamma_2\Phi)\sqrt{1 - \frac{(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)^2 + (\dot{v}_{g,2})^2}{c_g^2}}, \quad (6.36)$$
$$\eta = \delta\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g, \quad (6.37)$$

[5] The term $\cos(\theta)$ in (6.32) and (6.36) can often probably be well approximated by the value 1 in the implementation (as is done in the linearized analysis below). This was pointed out by Dr. P. Ögren, FOI.

where $\gamma_1 > 0, \gamma_2 < 0$ and $\delta > 0$ are some constants (to be further discussed below).

Remark: By its symmetry around 0 for η, the control law (6.36), (6.37) is only applicable for non aggressive maneuvering, i.e. for cases where the aircraft will never roll "upside down" in order to exploit the airframe's capability to generate a higher positive normal acceleration in the body frame B.

Closed loop system

The closed loop system resulting from (6.29), (6.32) and (6.34), (6.35) is

$$\frac{d}{dt}(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g) = g\frac{c_g}{V}\sqrt{1-\frac{(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)^2+(\dot{v}_{g,2})^2}{c_g^2}}\big(-\delta\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g+\gamma_1\dot{v}_{g,2}^2+\gamma_2\dot{v}_{g,2}\Phi\big), \tag{6.38}$$

$$\frac{d}{dt}\dot{v}_{g,2} = g\frac{c_g}{V}\sqrt{1-\frac{(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)^2+(\dot{v}_{g,2})^2}{c_g^2}}\big(-\Phi-\gamma_1\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g\dot{v}_{g,2}-\gamma_2\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g\Phi\big), \tag{6.39}$$

$$\frac{d}{dt}\Phi = g\frac{c_g}{V}\sqrt{1-\frac{(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)^2+(\dot{v}_{g,2})^2}{c_g^2}}\big(\gamma_1\dot{v}_{g,2}+\gamma_2\Phi\big). \tag{6.40}$$

For $(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g, \dot{v}_{g,2}, \Phi) \in \mathcal{C}\times\mathbb{R}$ this system has a unique equilibrium point at $(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g, \dot{v}_{g,2}, \Phi) = (0,0,0)$ which is shown to be semi-globally asymptotically stable in Proposition 6.1 below.

Linearized closed loop dynamics

The system (6.38)–(6.40) is weakly (quadratically) nonlinear and the linearization around $(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g, \dot{v}_{g,2}, \Phi) = (0,0,0)$ is

$$\frac{d}{dt}(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g) = -g\frac{c_g}{V}\delta\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g, \tag{6.41}$$

$$\frac{d}{dt}\dot{v}_{g,2} = -g\frac{c_g}{V}\Phi, \tag{6.42}$$

$$\frac{d}{dt}\Phi = g\frac{c_g}{V}(\gamma_1\dot{v}_{g,2}+\gamma_2\Phi). \tag{6.43}$$

Thus, in the linearized equations (6.41)–(6.43) the dynamics for $\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g$ are decoupled from the dynamics of the $(\dot{v}_{g,2}, \Phi)$-subsystem and the dynamics for $\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g$ are given by a simple first order linear system with time constant

$$\frac{V}{gc_g\delta}.$$

The $(\dot{v}_{g,2}, \Phi)$-subsystem in (6.42), (6.43) in can be written

$$\frac{d}{dt}\begin{bmatrix}\dot{v}_{g,2}\\ \Phi\end{bmatrix} = g\frac{c_g}{V}\begin{bmatrix}0 & -1\\ \gamma_1 & \gamma_2\end{bmatrix}\begin{bmatrix}\dot{v}_{g,2}\\ \Phi\end{bmatrix} \tag{6.44}$$

where the system matrix has eigenvalues

$$\lambda_{1,2} = \frac{gc_g}{2V}\big(\gamma_2 \pm \sqrt{\gamma_2^2-4\gamma_1}\big). \tag{6.45}$$

The system in (6.44) is asymptotically stable at $(0,0)$ if and only if both eigenvalues λ_1, λ_2 lie strictly inside the left half plane in $\mathbb{C}$, i.e. if and only if

$$\gamma_1 > 0 \quad \text{and} \quad \gamma_2 < 0. \tag{6.46}$$

To gain some insight into how the two parameters γ_1 and γ_2 should be chosen we note that the first factor on the right of (6.36) can be written

$$g\frac{c_g}{V}\gamma_1(\dot{v}_{g,2}+\frac{\gamma_2}{\gamma_1}\Phi).$$

Thus, in the control law (6.36) one can view γ_1 as a gain parameter and the γ_2/γ_1 as a parameter defining the relative influence of $\dot{v}_{g,2}$ and Φ, respectively. With this formulation the ratio γ_2/γ_1 also determines the equilibrium point for the Φ-dynamics in (6.43).

Stability of the command generator control law

The stability analysis of the command generator control law will be based on a Lyapunov argument. As Lyapunov function (candidate) we will consider $\mathcal{V}:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to[0,\infty)$ given by

$$\mathcal{V}(x,y,z)=\frac{x^2}{2}+\frac{y^2}{2}+ayz+b\frac{z^2}{2} \tag{6.47}$$

where $b>a^2>0$. This guarantees that $\mathcal{V}$ is (strictly) positive definite but the constants a,b will be further specified below.

Along a solution trajectory to (6.29), (6.34), (6.35) near any point where $\cos(\theta)>0$ (cf. (6.32)) we have

$$\begin{aligned}\frac{d}{dt}&\mathcal{V}(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g,\dot{v}_{g,2},\Phi)=\\ &\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g\frac{d}{dt}(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)+\dot{v}_{g,2}\ddot{v}_{g,2}+a\ddot{v}_{g,2}\Phi+a\dot{v}_{g,2}\dot{\Phi}+b\Phi\dot{\Phi}=\\ &-g\frac{c_g}{V}\left(\eta\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g+\dot{v}_{g,2}\Phi+a\Phi^2\right)\sqrt{1-\frac{(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)^2+(\dot{v}_{g,2})^2}{c_g^2}}\\ &\qquad\qquad+p^{(W)}(-a\Phi\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g+a\dot{v}_{g,2}+b\Phi)\end{aligned}$$

and in closed loop (with (6.36), (6.37))

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g,\dot{v}_{g,2},\Phi)&=-g\frac{c_g}{V}\sqrt{1-\frac{(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)^2+(\dot{v}_{g,2})^2}{c_g^2}}\cdot\\ &\big(\delta(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)^2+\dot{v}_{g,2}\Phi+a\Phi^2-(\gamma_1\dot{v}_{g,2}+\gamma_2\Phi)(-a\Phi\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g+a\dot{v}_{g,2}+b\Phi)\big)=\\ &-g\frac{c_g}{V}\sqrt{1-\frac{(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)^2+(\dot{v}_{g,2})^2}{c_g^2}}\mathcal{W}(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g,\dot{v}_{g,2},\Phi;\gamma_1,\gamma_2,a,b),\end{aligned} \tag{6.48}$$

where $\mathcal{W}(\,\cdot\,,\,\cdot\,,\,\cdot\,;\gamma_1,\gamma_2,a,b):\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is the (continuous) parametrized function

$$\begin{aligned}\mathcal{W}(&\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g,\dot{v}_{g,2},\Phi;\gamma_1,\gamma_2,a,b)=\\ &\delta(\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g)^2-\gamma_1a(\dot{v}_{g,2})^2+(a+\gamma_2(a\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g-b))\Phi^2\\ &\qquad(1+\gamma_1(a\boldsymbol{\sigma}^T\dot{\boldsymbol{v}}_g-b)-\gamma_2a)\dot{v}_{g,2}\Phi.\end{aligned} \tag{6.49}$$

Thus, since $\mathcal{W}(\,\cdot\,,\,\cdot\,,\,\cdot\,;\gamma_1,\gamma_2,a,b)$ vanishes at $\mathbf{0}$ we see that any set $\mathcal{D}\subset\mathcal{C}\times\mathbb{R}$ (with $\mathcal{C}$ as in (6.33)) containing the origin $\mathbf{0}\in\mathbb{R}^3$ in its interior and for which we can show that $\mathcal{W}(\,\cdot\,,\,\cdot\,,\,\cdot\,;\gamma_1,\gamma_2,a,b)$ is positive definite in $\mathcal{D}\backslash\{\mathbf{0}\}$ is a positively invariant set for the closed loop dynamics (6.29), (6.32) and (6.34)–(6.37) (so that, in particular, $\cos(\theta)$ remains positive there). More precisely, we have the following result.

Proposition 6.1 *If $\delta > 0, \gamma_1 > 0, \gamma_2 < 0$ then the origin $\mathbf{0}$ on the state manifold $\mathcal{C} \times \mathbb{R}$ is an asymptotically stable equilibrium point for the (closed loop) dynamical system $\mathcal{S}$ given by* (6.29), (6.32) *and* (6.34), (6.35), *viz.*[6]

$$\begin{aligned}
\frac{d}{dt}(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g) &= -\eta g \frac{c_g}{V}\sqrt{1 - \frac{(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g)^2 + (\dot{v}_{g,2})^2}{c_g^2}} + p\dot{v}_{g,2}, \\
\frac{d}{dt}\dot{v}_{g,2} &= -\Phi g \frac{c_g}{V}\sqrt{1 - \frac{(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g)^2 + (\dot{v}_{g,2})^2}{c_g^2}} - p\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \\
\frac{d}{dt}\Phi &= p,
\end{aligned}$$

where the controls η, p are given by (6.36), (6.37). *The domain of attraction of $\mathbf{0}$ contains the (open) ellipsoid*

$$\mathcal{E}(c_g) = \{[\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \dot{v}_{g,2}, \Phi]^T \in \mathbb{R}^3 \mid \frac{(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g)^2}{2} + \frac{(\dot{v}_{g,2})^2}{2} + \frac{1}{\gamma_1}\frac{\Phi^2}{2} < c_g^2\}. \tag{6.50}$$

Proof: We can write $\mathcal{W}(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \dot{v}_{g,2}, \Phi; \gamma_1, \gamma_2, a, b)$ in (6.49) as the value of a quadratic looking form

$$\mathcal{W}(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \dot{v}_{g,2}, \Phi; \gamma_1, \gamma_2, a, b) = \begin{bmatrix} \boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g \\ \dot{v}_{g,2} \\ \Phi \end{bmatrix}^T \mathcal{Q}(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \gamma_1, \gamma_2, a, b) \begin{bmatrix} \boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g \\ \dot{v}_{g,2} \\ \Phi \end{bmatrix} \tag{6.51}$$

where $\mathcal{Q}(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \gamma_1, \gamma_2, a, b) \in \mathbb{R}^{3\times 3}$ is the symmetric matrix

$$\mathcal{Q}(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \gamma_1, \gamma_2, a, b) = \begin{bmatrix} \delta & 0 & 0 \\ 0 & -\gamma_1 a & \frac{1}{2}(1 + \gamma_1(a\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g - b) - \gamma_2 a) \\ 0 & \frac{1}{2}(1 + \gamma_1(a\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g - b) - \gamma_2 a) & a + \gamma_2(a\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g - b) \end{bmatrix}. \tag{6.52}$$

The matrix $\mathcal{Q}(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \gamma_1, \gamma_2, a, b)$ is positive definite if and only if $\delta > 0$ and the lower right 2×2 sub matrix is positive definite. By a completing-the-squares argument it is easy to see that the lower right 2×2 sub matrix of $\mathcal{Q}(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \gamma_1, \gamma_2, a, b)$ is positive definite if and only if

$$-\gamma_1 a > 0 \tag{6.53}$$

and

$$a + \gamma_2(a\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g - b) > -\frac{1}{4\gamma_1 a}(1 + \gamma_1(a\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g - b) - \gamma_2 a)^2. \tag{6.54}$$

A necessary and sufficient condition[7] for the (continuous) function $\mathcal{W}$ in (6.49) and (6.51) to be positive definite in a neighborhood of $\mathbf{0}$ is that the matrix

[6] We consider the airspeed V as a constant here.

[7] From the analysis of the linearized system in (6.41)–(6.43) and a well-known result about stability and quadratic Lyapunov functions for linear systems cf. e.g. (Sastry, 1999, Thm. 5.34, 5.36), we know that the conditions (6.53) and (6.54) for $\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g = 0$ are necessary and sufficient for asymptotic stability of the closed loop system (6.38)–(6.40) in some neighborhood of $\mathbf{0}$.

$\mathcal{Q}(0, \gamma_1, \gamma_2, a, b)$ is positive definite, i.e. that (6.53) and (6.54) hold for $\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g = 0$.

We are interested in positive definiteness of $\mathcal{W}$ under the additional condition that $b > a^2 > 0$ so that $\mathcal{V}$ in (6.47) is indeed a Lyapunov function. The conditions (6.53) and (6.54) hold for $\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g = 0$ and $b > a^2 > 0$ precisely when the conditions in (6.46) hold, i.e. when $\gamma_1 > 0, \gamma_2 < 0$. To see that the conditions in (6.46) are necessary for the conditions (6.53) and (6.54) to hold for $\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g = 0$ and $b > a^2 > 0$ we can proceed as follows. To begin with, note that when $\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g = 0$ the inequality (6.54) is equivalent to

$$(1 - \gamma_1 b + \gamma_2 a)^2 + 4\gamma_1 a^2 - 4\gamma_2 a < 0, \tag{6.55}$$

Then, assume (in order to reach a contradiction) that $\gamma_1 \leq 0$ and that (6.53) and (6.55) hold for $b > a^2 > 0$. However, when $\gamma_1 \leq 0$ and $b > a^2$ we have $\gamma_1 a^2 \geq \gamma_1 b$ and hence

$$\begin{aligned}
(1 - \gamma_1 b + \gamma_2 a)^2 &+ 4\gamma_1 a^2 - 4\gamma_2 a = \\
&1 - 2\gamma_1 b + 2\gamma_2 a + \gamma_1^2 b^2 - 2\gamma_1\gamma_2 ab + \gamma_2^2 a^2 + 4\gamma_1 a^2 - 4\gamma_2 a \geq \\
&1 - 2\gamma_1 b + 2\gamma_2 a + \gamma_1^2 b^2 - 2\gamma_1\gamma_2 ab + \gamma_2^2 a^2 + 4\gamma_1 b - 4\gamma_2 a = \\
&1 + 2\gamma_1 b - 2\gamma_2 a + \gamma_1^2 b^2 - 2\gamma_1\gamma_2 ab + \gamma_2^2 a^2 = (1 + \gamma_1 b - \gamma_2 a)^2 \geq 0,
\end{aligned}$$

which contradicts (6.55). Hence, we must have $\gamma_1 > 0$ in order for (6.53) and (6.54) hold for $\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g = 0$ and $b > a^2$. Moreover, it follows that we must have $a < 0$. Assume instead (again in order to reach a contradiction) that $\gamma_2 \geq 0$ and that (6.53) and (6.55) hold for $\gamma_1 > 0$. The inequality (6.55) implies that $\gamma_1 a^2 - \gamma_2 a < 0$ and the inequality (6.53) implies that $a < 0$ but then $\gamma_1 a^2 - \gamma_2 a > 0$ which contradicts (6.55). Hence, also $\gamma_2 < 0$ in order for (6.53) and (6.54) hold for $\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g = 0$ and $b > a^2 > 0$.

In view of the above, we will henceforth only consider the case $\gamma_1 > 0, \gamma_2 < 0$ and $a < 0$. For this case, the two inequalities (6.53) and (6.54) are equivalent to the single inequality

$$\big(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g + \frac{1}{\gamma_1 a}(1 - \gamma_1 b + \gamma_2 a)\big)^2 + \frac{4}{\gamma_1 a}(a - \frac{\gamma_2}{\gamma_1}) < 0. \tag{6.56}$$

We know from the above that a necessary and sufficient condition for the function $\mathcal{W}$ in (6.49) to be positive definite in a neighborhood of $\mathbf{0}$ is that the inequality (6.54) has a solution for $\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g = 0$, i.e. that (6.56) has a solution for $\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g = 0$. Such a solution can only exist if the last term on the left in (6.56) is negative, i.e. if

$$a \in (\frac{\gamma_2}{\gamma_1}, 0). \tag{6.57}$$

When this holds, the set of values $\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g$ for which (6.56) is satisfied becomes the interval (ℓ_1, ℓ_2) given by

$$\ell_{1,2} = -\frac{1}{\gamma_1 a}(1 - \gamma_1 b + \gamma_2 a) \pm 2\sqrt{-\frac{1}{\gamma_1 a}(a - \frac{\gamma_2}{\gamma_1})}. \tag{6.58}$$

The interval (ℓ_1, ℓ_2) becomes symmetric about 0 (and in particular contains the point 0) when

$$1 - \gamma_1 b + \gamma_2 a = 0 \tag{6.59}$$

so we must ascertain that this manifold in $(\gamma_1, \gamma_2, a, b)$-space contains a point where $b > a^2$. However, on the manifold defined by (6.59) we always have $b > a^2$ when (6.57) holds since then

$$b - a^2 = \frac{1 + \gamma_2 a}{\gamma_1} - a^2 = \frac{1}{\gamma_1} - a\big(a - \frac{\gamma_2}{\gamma_1}\big) > 0. \tag{6.60}$$

Thus, when (6.57) holds the interval (ℓ_1, ℓ_2) can be made arbitrarily large by making the constant $a < 0$ close to 0 and then it can be centered, so that it contains 0, by selecting b so that (6.59) holds.

Summing up, when $\gamma_1 > 0, \gamma_2 < 0$, the constant $a < 0$ satisfies (6.57) and the constant $b > 0$ is selected so that (6.59) holds the inequality (6.60) is satisfied and the interval (ℓ_1, ℓ_2) of values of $\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g$ that satisfy (6.53) and (6.54) is given by (6.58) and contains the point 0. Thus, the function $\mathcal{V}$ in (6.47) is then admissible as a Lyapunov function (candidate) and the matrix $\mathcal{Q}(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \gamma_1, \gamma_2, a, b)$ in (6.52) is positive definite for $\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g$ in the interval (ℓ_1, ℓ_2) so that the function $\mathcal{W}$ in (6.49) and (6.51) is positive definite on the set

$$\{(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \dot{v}_{g,2}, \Phi) \in \mathbb{R}^3 \,|\, (\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \dot{v}_{g,2}) \in \mathcal{C}, \boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g \in (\ell_1, \ell_2)\}. \tag{6.61}$$

Since the constant $a < 0$ can always be chosen arbitrarily close to 0 the arguments above show that for $\gamma_1 > 0, \gamma_2 < 0$ there exist parameter values a, b such that $\mathcal{V}$ is locally well defined as Lyapunov function for the system $\mathcal{S}$. Moreover, the interval (ℓ_1, ℓ_2) in (6.58) can always be made large enough that $\mathcal{C} \cap ((\ell_1, \ell_2) \times \mathbb{R}) = \mathcal{C}$ (which happens when $-\ell_1, \ell_2 \geq c_g^2$). Hence, for $a < 0$ sufficiently close to zero, any level set of $\mathcal{V}$ contained in the cylinder $\mathcal{C} \times \mathbb{R}$ is part of the domain of attraction of the equilibrium at $\mathbf{0}$ for the system $\mathcal{S}$ (cf. e.g. Sastry (1999, Prop. 5.22)). The only remaining point is to show that the domain of attraction also includes any ellipsoid of the form (6.50).

Given $\gamma_1 > 0, \gamma_2 < 0$ and a pair $a < 0, b > 0$ such that (6.57), (6.59) are fulfilled, any (open) level set

$$\{[x, y, z]^T \in \mathbb{R}^3 \,|\, \mathcal{V}(x, y, z) < c^2\}$$

where $c > 0$, is an (open) ellipsoid $\mathcal{E}_a(c)$ of the form

$$\mathcal{E}_a(c) = \{[x, y, z]^T \in \mathbb{R}^3 \,|\, \frac{x^2}{2} + \frac{y^2}{2} + ayz + \frac{1 + \gamma_2 a}{\gamma_1} \frac{z^2}{2} < c^2\}.$$

We have $\mathcal{E}_0(c) \subseteq \mathcal{C} \times \mathbb{R}$ if and only if $c \leq c_g$ and when this holds $\mathrm{dist}(\mathcal{E}_0(c), (\mathcal{C} \times \mathbb{R})^c) = c_g - c$. We now claim that for $c < c_g$ there exists $a < 0$ such that (6.57) is fulfilled and

$$\mathcal{E}_0(c) \subset \mathcal{E}_a(c + (c_g - c)/2) \subset \mathcal{C} \times \mathbb{R}. \tag{6.62}$$

However, this is clear since, for fixed $c < c_g$, the functions $a \mapsto \mathrm{dist}(\mathcal{E}_a(c + (c_g - c)/2), (\mathcal{C} \times \mathbb{R}))^c$ and $a \mapsto \mathrm{dist}(\mathcal{E}_0(c)), (\mathcal{E}_a(c + (c_g - c)/2))^c$ are continuous and for $a = 0$ the relations (6.62) are satisfied (with strictly positive distances). Thus, the domain of attraction of the equilibrium at $\mathbf{0}$ for the system $\mathcal{S}$ includes $\mathcal{E}_0(c)$ for any $c < c_g$. However, since $\mathcal{E}(c_g) = \mathcal{E}_0(c_g)$ the statement in the proposition about the domain of attraction in (6.50) follows. $\square$

Remark: The value of the constant $\gamma_2 < 0$ does not affect the domain of attraction but it greatly effects the dynamics.

Remark: One can also consider more complex control laws that will cancel more terms in the Lyapunov function derivative (6.48). For instance, one can instead of (6.37) consider the control law

$$\eta = \boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g (\delta - \gamma_1 a \dot{v}_{g,2} \Phi - \gamma_2 a \Phi^2).$$

The matrix $\mathcal{Q}(\boldsymbol{\sigma}^T \dot{\boldsymbol{v}}_g, \gamma_1, \gamma_2, a, b)$ in (6.52) in the stability proof would then be replaced by $\mathcal{Q}(0, \gamma_1, \gamma_2, a, b)$, which is the same as for the linearization. Hence, the stability region would not be enlarged. Moreover, the linearization of the closed loop system would be the same as in (6.41)–(6.43) so the closed loop dynamics would be essentially unchanged.

A Relative motion

A.1 Relative velocity

Let $\boldsymbol{x}(t) \in \mathbb{R}^3$ be a continuously time differentiable vector and let $\boldsymbol{R}(t) \in \mathbb{R}^{3\times 3}$ be a rotation matrix, i.e. $\boldsymbol{R}(t)^T = \boldsymbol{R}(t)^{-1}$, $\det(\boldsymbol{R}(t)) = 1$, with continuously time differentiable elements, where time t is defined in some open interval $\mathcal{I} \subset \mathbb{R}$. Define the vector $\boldsymbol{X}(t) \in \mathbb{R}^3$ by

$$\boldsymbol{X}(t) = \boldsymbol{R}(t)\boldsymbol{x}(t), \quad t \in \mathcal{I}. \tag{A.1}$$

Then

$$\dot{\boldsymbol{X}}(t) = \dot{\boldsymbol{R}}(t)\boldsymbol{x}(t) + \boldsymbol{R}(t)\dot{\boldsymbol{x}}(t) = \dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T\boldsymbol{R}(t)\boldsymbol{x}(t) + \boldsymbol{R}(t)\dot{\boldsymbol{x}}(t) = \\ \dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T\boldsymbol{X}(t) + \boldsymbol{R}(t)\dot{\boldsymbol{x}}(t). \tag{A.2}$$

Since

$$0 = \frac{d}{dt}\boldsymbol{I}_3 = \frac{d}{dt}\big(\boldsymbol{R}(t)\boldsymbol{R}(t)^T\big) = \dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T + \boldsymbol{R}(t)\dot{\boldsymbol{R}}(t)^T$$

it follows that $\dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T$ is skew-symmetric; $(\dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T)^T = -\dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T$. A skew-symmetric linear map $\mathbb{R}^3 \to \mathbb{R}^3$ can be identified with a cross product via the representation $\boldsymbol{y} \times \boldsymbol{z} = \boldsymbol{S}(\boldsymbol{y})\boldsymbol{z}$ where

$$\boldsymbol{S}(\boldsymbol{y}) = \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix}, \quad \boldsymbol{y} = [y_1, y_2, y_3]^T,$$

and we thus have $\boldsymbol{S}(\boldsymbol{y})^T = -\boldsymbol{S}(\boldsymbol{y}) = -\boldsymbol{y} \times \cdot$. It follows that there exists (for each t) a uniquely defined solution $\boldsymbol{\Omega}(t) \in \mathbb{R}^3$ to the following equation (to hold identically for all $\boldsymbol{z} \in \mathbb{R}^3$)

$$\boldsymbol{\Omega}(t) \times \boldsymbol{z} = \dot{\boldsymbol{R}}(t)\boldsymbol{R}(t)^T\boldsymbol{z}. \tag{A.3}$$

Using $\boldsymbol{\Omega}(t)$ defined by (A.3) the relation (A.2) can be written

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{\Omega}(t) \times \boldsymbol{X}(t) + \boldsymbol{R}(t)\dot{\boldsymbol{x}}(t),$$

and if we define the vector $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ by

$$\boldsymbol{\Omega}(t) = \boldsymbol{R}(t)\boldsymbol{\omega}(t), \quad t \in \mathcal{I},$$

we can further write

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{R}(t)\big(\boldsymbol{\omega}(t) \times \boldsymbol{x}(t) + \dot{\boldsymbol{x}}(t)\big), \tag{A.4}$$

where we have used the fact that rotations preserve angles and therefore commute with the cross product operation (cf. e.g. (Arnold, 1989, p. 131)).

Bibliography

Ananthkrishnan, N. & Unnikrishnan, S. 2001 Literal approximations to aircraft dynamic modes. *J. Guidance, Control and Dynamics*, **24**(6), 1196–1203.

Arnold, V. 1989 *Mathematical Methods of Classical Mechanics*. Graduate Texts in Mathematics. Springer-Verlag, New York, NY, second edition.

Field, E. 1993 The application of a C* flight control law to large civil transport aircraft. Technical Report 9303, College of Aeronautics, Cranfield Inst. Technol., Cranfield, Bedford, England.

Forssell, L. & Nilsson, U. 2005 ADMIRE the aero-data model in a research environment version 4.0, model description. Technical Report FOA-R–1624–SE, Swedish Defence Research Agency, FOI.

Goman, M. & Khramstovsky, A. 1997 Global stability analysis of nonlinear aircraft dynamics. In *Proc. AIAA Atmospheric Flight Mechanics Conference*, pages 662–672. New Orleans, LA. AIAA paper 97-3721.

Goman, M., Khramtsovsky, A. & Kolesnikov, E. 2007 Investigations of the ADMIRE manoeuvering capabilities using qualitative methods. In Bates, D. & Hagström, M. (Editors), *Nonlinear Analysis and Synthesis Techniques for Aircraft Control*, volume 365, pages 301–324. Springer-Verlag, Berlin, Germany.

Hodgkinson, J. 1999 *Aircraft Handling Qualities*. Education Series. AIAA, Reston, VA.

Jahnke, C. 1998 On the roll-coupling instabilities of high performance aircraft. *Phil. Trans. Roy. Soc. Lond. A*, **356**, 2223–2239.

Johansson, H. 1998 Mathematical modeling and simulation of a missile. Technical Report FOA-R–98-00876-314–SE, Swedish Defence Research Establishment, FOA.

MIL-HDBK-1797 1997 Flying qualitites of piloted aircraft. US Department of Defense. Handbook MIL-HDBK-1797 (superseding MIL-STD-1797A).

Phillips, W. 2010 *Mechanics of Flight*. Wiley, Hoboken, NJ, second edition.

Raymer, D. 2006 *Aircraft Design: A Conceptual Approach*. Education Series. AIAA, Reston, VA, fourth edition.

Robinson, J. 2010 Flight mechanical modeling of a precision guided mortar munition. Technical Report FOI-R–2618–SE, Swedish Defence Research Agency, FOI.

Salling, E., Strömbäck, P., Lindh, J., Hildings, C., Wallin, N. & Fredriksson, M. 2009 MERLIN software architecture document. Technical Report FOI-R–2486–SE, Swedish Defence Research Agency, FOI.

Sastry, S. 1999 *Nonlinear Systems: Analysis, Stability and Control*, volume 10 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, NY.

Stevens, B. & Lewis, F. 2003 *Aircraft Control and Simulation*. John Wiley & Sons, Inc., New York, NY, second edition.

Tobie, H., Elliott, E. & Malcom, L. 1966 A new longitudinal handling qualities criterion. In *Proc. National Aerospace Electronics Conf.*, pages 93–99. Dayton, OH.