Topology optimization: a literature review

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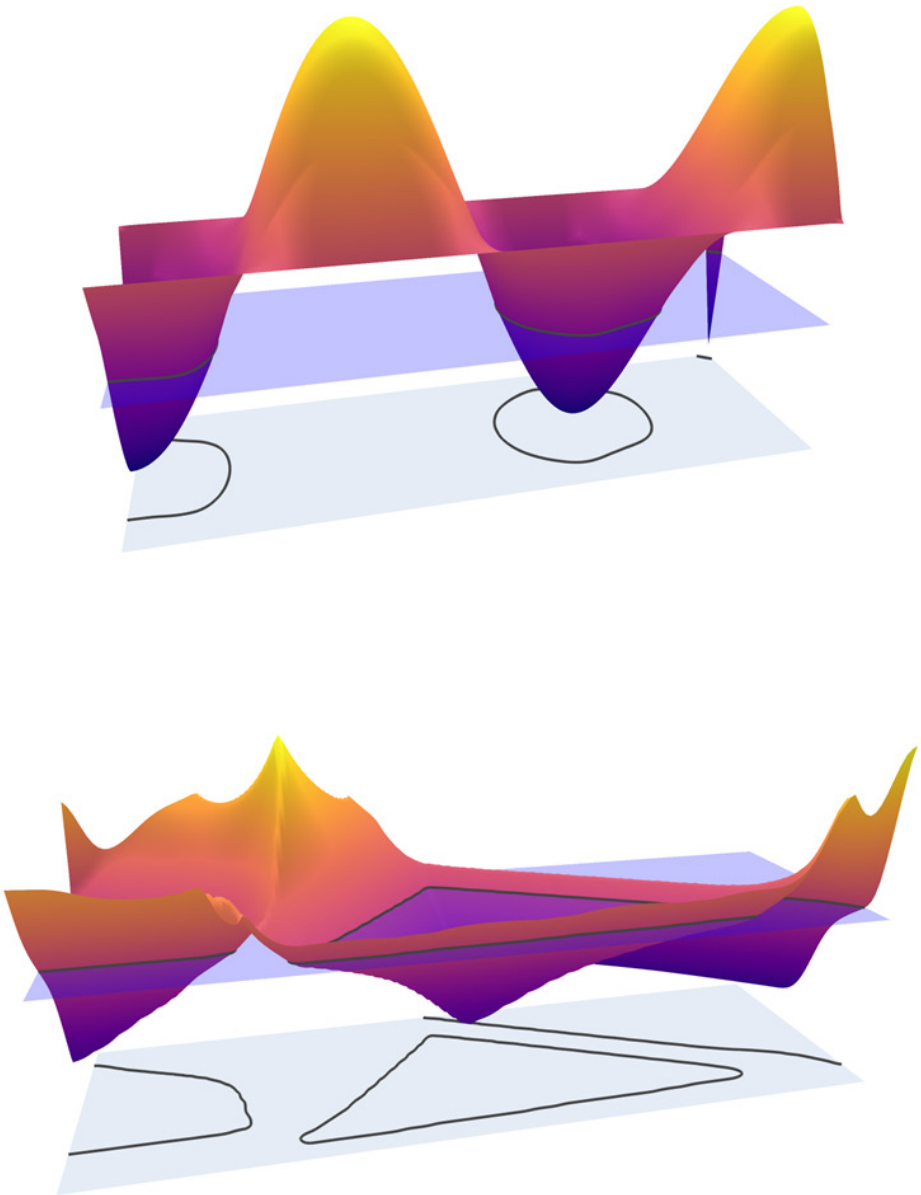

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Summary

Topology optimization has become an important part of the structural design process; by finding the optimal layout of a material in a domain, subject to some constraint, an informed initial design can be found early on and potentially save both time and costs in the iterative design procedure. The field has continuously evolved by developing different numerical methods and approaches, as well as in its application to an increasingly wider range of physical disciplines.

The aim of this document is to give an introduction to the field of topology optimization by giving a basic overview of the main methods and approaches used as well as providing an up-to-date review of recent literature concerning physical disciplines with applications of topology optimization that can be relevant to aircraft design, namely elastic structures possibly involving multi-scale or anisotropic material properties, fluid-structure interaction, and radar cross section properties. In addition we present a list of available open-source software that either implement a general approach using some of the well-used methods in topology optimization or a more specific approach connected to a particular study.

Keywords: topology optimization, structural optimization

Sammanfattning

Topologioptimering har blivit en viktig del av den strukturella designprocessen; genom att hitta den optimala layouten för ett material i en domän, under vissa bivillkor, kan en informerad initial design hittas tidigt och potentiellt spara både tid och kostnader i den iterativa designprocessen. Området har kontinuerligt utvecklats till att omfatta olika numeriska metoder och tillvägagångssätt, såväl som i dess tillämpning på ett ökande antal fysiska discipliner. Syftet med detta dokument är att ge en introduktion till området topologioptimering genom att ge en grundläggande översikt av de huvudsakliga metoderna och tillvägagångssätten som används, samt att ge en uppdaterad genomgång av aktuell litteratur beträffande fysiska discipliner med tillämpningar av topologioptimering som kan vara relevanta för flygteknik, nämligen elastiska strukturer, flerskaliga eller anisotropa materialstrukturer, interaktion mellan strömning och struktur, och reduktion av radarmålytor. Som ett delresultat av granskingen presenteras en lista av tillgänglig programvara med öppen källkod som antingen implementerar ett allmänt tillvägagångssätt med några av de välanvända metoderna för topologioptimering eller ett mer specifikt tillvägagångssätt kopplat till en viss studie.

Nyckelord: topologioptimering, strukturoptimering

Table of contents

Nomenclature

Abbreviations

AM	Additive Manufacturing
CAD	Computer-Aided Design
DBM	Density-Based Method
DDF	Density Distribution Function
EGM	Explicit Geometry Method
ESO/BESO	Evolutionary Structural Optimization/Bi-directional ESO
FDM	Finite Difference Method
FE(M)	Finite Element (Method)
FSI	Fluid-Structure Interaction
FVM	Finite Volume Method
GPM	Geometry Projection Method
IBM	Immersed Boundary Method
IGA	Isogeometric Analysis
LSM	Level Set Method
ML	Machine Learning
MMA	Method of Moving Asymptotes
MMC/MMV	Moving Morphable Components/Voids
MS	Metasurfaces
NS	Navier-Stokes
NURBS	Non-Uniform Rational B-Splines
OC	Optimality Criteria
PDE	Partial Differential Equation
PFM	Phase-field Method
RAMP	Rational Approximation of Material Properties
RBF	Radial Basis Function
RCS(R)	Radar Cross Section (Reduction)
SIMP	Solid Isotropic Material with Penalization

SO	Shape Optimization
TO	Topology Optimization
TOBS	TO of Binary Structures
VEM	Virtual Element Method
XFEM	Extended Finite Element Method

Greek letters

ε	Strain-rate tensor
λ	Adjoint variable
ρ	Material (pseudo) density, at each point in $\mathscr{D}$
σ	Cauchy stress tensor
ϕ	Level set function
Ω	Material domain, subset of $\mathscr{D}$ which consists of non-void material

Latin letters

$\mathscr{C}$	Stiffness tensor
$\mathscr{D}$	Design domain
$\mathscr{F}$	Set of admissible or possible designs
F	State equation (partial differential operator) to which u is a solution
g_i, h_i	Inequality and equality constraints
$\tilde{g}_i, \tilde{h}_i$	Inequality and equality constraints in the reduced problem
J ($\widetilde{J}$)	(Reduced) Objective functional, the function to be minimized in the optimization problem
m	Design variable
p	Penalty factor (in material interpolation function)
$\mathbb{R}$	Real numbers
u	State variable, e.g., displacement of structure

1 Introduction

Topology Optimization (TO) is a physics-based class of design optimization methods concerned with the problem of finding the best layout of materials in a domain according to specified criteria [1, 2]. The main objective of structural TO is to find the distribution of a material, within a specified design domain and subject to boundary conditions and loads, typically corresponding to the minimal flexibility of the structure, also called compliance.

Based on theoretical results for the optimal mass distribution in beam structures [3, 4], Prager & Rozvany [5] presented the basis for the theory of topology optimization. However, for more general scenarios, when there is no knowledge a priori on the existence of analytical solutions, the use of numerical approaches becomes necessary. The study by Bendsøe & Kikuchi [6], in which the authors developed a homogenization method to solve structural optimization problems, is considered to be the starting point for numerical topological optimization. Subsequently, several numerical methods have been developed, adapted and successfully applied to topological optimization problems for a range of different physics. For example, topology optimization has been used in the aerospace industry to design the internal structure of aircraft pylons [7], wings [8–10] and fuselage [11], for component designs as landing gear [12], nacelle hinge brackets [13] and heat exchanger fins [14], and to improve electromagnetic performance of structures by minimizing the radar cross section of surfaces [15].

The objective of this paper is to give an overview of the most prominent methods. It provides, additionally, a list of open-source software that implements these, as well as a review of recent studies that focus on aeronautical engineering.

The study is structured as follows: the rest of chapter 1 states the general optimization and topology optimization problems, chapter 2 gives an overview of the most commonly used methods in TO, chapter 3 provides a review of recent literature by area of application, and chapter 4 presents a collection of open-source software.

1.1 Design optimization problem and reduced formulation

1.1.1 From design specifications to optimization

The purpose of physics-based optimization in product development is to accelerate the discovery of designs which satisfy a set of specifications and the laws of physics.

The specifications are amenable to optimization if they can be translated to an optimization problem which is generally defined by selecting the variables of design, a cost function to minimize, and constraints. Among the specifications, some contribute to the definition of the cost function while others contribute to the definition of equality or inequality constraints. [1]

In addition, we must take care that adequate physical laws are satisfied, which gives rise to additional constraints taking the form of *state equations*. An example arises in the design of a structure made of an isotropic metallic material. If it is supposed to undergo only small deformations when subject to loads and boundary conditions, it is common to assume that it obeys the principles of linear elastic theory. In that case, the physical laws introduce constraints on the *state variable* which describes the deformation of the loaded structure and these constraints, the state equations, take the form of Partial Differential Equations (PDE). The state equations depend on the underlying physics of the design problem and can describe other physics than structural elasticity. For non-trivial problems of design, the state

[1] There are no rules dictating how to define objectives and constraints, more than that the optimization problem must be well-defined and unambiguous.

equations are written as $F(u,m) = 0$, where F is the PDE operator, which depends on the *design variable* m and the state variable u, defining the state equations.

A large family of structural design problems [e.g., 1, 16] can thus be resolved from a set $\mathscr{F}$ of possible designs looking for an optimal m that minimizes the *objective function* J (also called the *cost function*), i.e., an m that satisfies the optimization problem:

$$\begin{aligned} \min_{m\in\mathscr{F}} \quad & J(u,m) \\ \text{subject to:} \quad & F(u,m) = 0, \quad u \in \mathscr{U} \\ & g_i(u,m) \leq 0, \quad i \in \mathscr{I}, \\ & h_j(u,m) = 0, \quad j \in \mathscr{E}, \end{aligned} \tag{1}$$

where $\mathscr{U}$ is the set containing admissible states (that is, states which are sufficiently regular satisfying essential boundary conditions), and $\mathscr{E}$ and $\mathscr{I}$ are the index sets of equality and inequality constraints, respectively.

1.1.2 Reduced optimization problem

Assuming that the state equations admit a unique u for any $m \in \mathscr{F}$, problem (1) can be reformulated as a *reduced problem* of optimization [17], also called *nested problem* [18]. This justifies using the notations $u(m)$ and $\widetilde{J}(m)$, where the latter is called the *reduced objective (cost) function*:

$$\begin{aligned} \min_{m\in\mathscr{F}} \quad & \widetilde{J}(m) \\ \text{subject to:} \quad & \tilde{g}_i(m) \leq 0, \quad i \in \mathscr{I}, \\ & \tilde{h}_j(m) = 0, \quad j \in \mathscr{E}, \end{aligned} \tag{2}$$

where $\tilde{J}(m) := J(u(m),m)$, $\tilde{g}_k(m) := g(u(m),m)$ and $\tilde{h}_k(m) := h(u(m),m)$ for all k.

1.2 Topology design optimization

In TO the design variable m controls the distribution of a material in the *design domain* $\mathscr{D}$. Typically, the design variable controls the material domain $\Omega = \Omega(m) \subset \mathscr{D}$ or a description of the material domain boundaries $\partial\Omega(m)$.

The reduced design optimization problem (2), in the context of TO, usually introduces a constraint on the volume of material $V_\Omega(m) = \int_{\Omega(m)} dx$ [19]:

$$\begin{aligned} \min_{m\in\mathscr{F}} \quad & \widetilde{J}(m) \\ \text{subject to:} \quad & g_0(m) = V_\Omega(m) - V_0 \leq 0, \\ & \tilde{g}_i(m) \leq 0, \quad i \in \mathscr{I}, \\ & \tilde{h}_j(m) = 0, \quad j \in \mathscr{E}, \end{aligned} \tag{3}$$

where V_0 is the upper limit for the volume of material and where the unique solution of state equation $u \in \mathscr{U}$ necessarily depends on $\Omega(m)$.

1.3 Solution process for the reduced problem

The resolution of reduced optimization problems like (2) requires to choose a method to solve the state equations, which for PDEs is a method of discretization, and to select an optimization strategy. The discretization methods are not discussed here but we give some details about the state-equation for linear elasticity in section 1.3.3.

Design problems like (2) or (3) are generally *nonlinear problems of optimization*, meaning that any of the functions, cost or constraints, may be nonlinear with respect to the design variables [20], even for simple structural sizing problems [18]. The resolution of those problems requires special algorithms, as presented in section 1.3.1, which are usually *gradient-based* algorithms. In the context of TO, the number of design variables can easily be thousands or millions, which requires a special strategy for the calculation of the gradients of the cost and constraints, as discussed in section 1.3.2.

1.3.1 Optimization algorithms

Algorithms for solving nonlinear optimization problems typically build a sequence of approximations $\mathscr{P}_0, \mathscr{P}_1, \ldots$ of the original optimization problem $\mathscr{P}$, such as (3), by for example using *quadratic-linear* approximations[2]. The principle is to define, at each iteration n, an approximative problem $\mathscr{P}_n$ whose solution provides the direction of an improved approximation m_n of the original design problem solution in the sense that m_n satisfies the constraints of $\mathscr{P}$ and improves the cost function in comparison to the previous estimation as $\widetilde{J}(m_n) < \widetilde{J}(m_{n-1})$.

For a discussion on the major sequential optimization algorithms used for structure design the reader may refer to [18]: these are the Sequential Linear Programming (SLP), Sequential Quadratic Programming (SQP) [21, 22], Convex Linearization (CONLIN) [23], and the Method of Moving Asymptotes (MMA)[3] [24, 25]. The merit of these methods lies in how well the sub-problems P_n approximate the original nonlinear problem. In particular, the subproblems in CONLIN and MMA are formulated considering the form of the underlying structural problem. Belonging to the same category, the Optimality Criteria (OC) method [e.g., 26, 27] is tailored specifically to the problem (3) and has been widely used due to its ease of implementation and computational efficiency.

For general constrained TO problems, possibly nonlinear or non-convex, nonlinear programming methods like interior-point methods [28–30] and SQP already cited above have proven to be successful, with the MMA [24, 25] being the most widely used. More recently, a Null Space Optimization method has been adapted and successfully applied to TO problems [31, 32]. In Rojas-Labanda *et al.* [33] numerical aspects of some of the above methods are compared.

The methods cited above are *gradient-based* algorithms, meaning that they require the first derivatives of the reduced cost and constraints, at least, which is further discussed in section 1.3.2 below.

1.3.2 Gradient calculation: adjoint state

The optimization algorithms cited previously need to calculate at least the first derivative of the reduced cost and constraints in problem (3), usually at each iteration, in order to build sequential approximations of the design optimization problem introduced in section 1.3.1. Whenever feasible, the calculation of the functional derivatives is most efficiently carried out by solving an *adjoint-state* equation, which is briefly summarized in this section.

Assuming that J is (Fréchet) differentiable, the linear sensitivity, or first order derivative, given by the total derivative $\frac{\mathrm{d}J}{\mathrm{d}m}$, is expressed as

$$\frac{\mathrm{d}J}{\mathrm{d}m} = \frac{\mathrm{d}\widetilde{J}}{\mathrm{d}m} = \frac{\partial J}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}m} + \frac{\partial J}{\partial m}. \tag{4}$$

[2] Also called quadratic programming: quadratic-linear optimization problems are characterized by a quadratic cost function and linear constraints, and they are the backbone of SQP.

[3] Codes for MMA and its global variant GCMMA are provided by the author at `https://www.smoptit.se` (accessed 2026-03-09). Other open-source implementations also exist.

The terms $\frac{\partial J}{\partial u}$ and $\frac{\partial J}{\partial m}$ can usually be calculated given that $J(u(m),m)$ is explicitly defined. Differentiating the state equation $F(u,m)=0$ gives a relation for $\frac{\mathrm{d}u}{\mathrm{d}m}$, which is only defined implicitly by the state u, as

$$\frac{\partial F}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}m} = -\frac{\partial F}{\partial m}. \tag{5}$$

Inserting the expression for $\frac{\mathrm{d}u}{\mathrm{d}m} = -\frac{\partial F}{\partial u}^{-1}\frac{\partial F}{\partial m}$ from eq. (5) into eq. (4) gives

$$\frac{\mathrm{d}J}{\mathrm{d}m} = -\frac{\mathrm{d}J}{\mathrm{d}u}\frac{\partial F}{\partial u}^{-1}\frac{\partial F}{\partial m} + \frac{\partial J}{\partial m}. \tag{6}$$

Equation (6) can be evaluated in two ways: either by directly solving eq. (5) for $\frac{\mathrm{d}u}{\mathrm{d}m}$ or by introducing the auxiliary variable $\lambda^* = \frac{\mathrm{d}J}{\mathrm{d}u}\frac{\partial F}{\partial u}^{-1}$ and solving the associated equation

$$\frac{\partial F}{\partial u}^{*}\lambda = \frac{\mathrm{d}J}{\mathrm{d}u}^{*}, \tag{7}$$

where the * superscript denotes the adjoint of a linear map. Equation (7) is called the *adjoint equation*, or model, for the adjoint variable λ, while eq. (5) is called the *tangent equation* for $\frac{\mathrm{d}u}{\mathrm{d}m}$. The adjoint solution λ allows the computation of the derivative of the cost function as

$$\frac{\mathrm{d}J}{\mathrm{d}m} = -\lambda^*\frac{\partial F}{\partial m} + \frac{\partial J}{\partial m}. \tag{8}$$

In numerical applications, the state, tangent and adjoint equations are discretized using, for example, the Finite Element Method (FEM); the discretization of the design in TO is the topic of chapter 2. In this case the linearized operators $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial u}^{*}$ can be represented by matrices and the adjoint (*) is the conjugate transpose. Let n_u and n_m be the dimensions of the discretized state u and design m, respectively. The dimensions of the terms in eqs. (5) and (7) are then

$$\underset{n_u\times n_u}{\frac{\partial F}{\partial u}}\ \underset{n_u\times n_m}{\frac{\mathrm{d}u}{\mathrm{d}m}} = -\underset{n_u\times n_m}{\frac{\partial F}{\partial m}} \quad \text{and} \quad \underset{n_u\times n_u}{\frac{\partial F}{\partial u}^{*}}\ \underset{n_u\times 1}{\lambda} = \underset{n_u\times 1}{\frac{\mathrm{d}J}{\mathrm{d}u}^{*}},$$

which reveals that the adjoint equation does not depend on n_m, i.e., the number of design variables, and, in contrast, that the right-hand side of the tangent equation is of dimension $n_u \times n_m$. Calculating the derivative of the cost function using the solution of the tangent equation eqs. (4) and (5) is thus n_m times more expensive than using the solution of the adjoint equation eqs. (7) and (8).

It is important to observe that if the constraints h_i or g_j depend on u, the optimization algorithm will require their gradient, which requires solving additional adjoint equations like eq. (7). Hence, increasing the number of additional constraints increases the cost of the adjoint formulation [e.g., 19, 34]. However, many TO problems (3) have many more n_m design variables than constraints which depend on the state u, making the use of the adjoint-based gradient calculation above very advantageous [19, 35].

Finally, for TO problems with many constraints, such as point-wise constraints on the state variable, or where a discrete binary (discontinuous) 0-1 design variable is used, gradient-free methods can be an efficient alternative to gradient-based algorithms [36, 37]. Some of these methods are discussed in the upcoming section.

1.3.3 State equation: linear elasticity

Since much of the original research on TO stems from optimization of structures, we present the state equations for the static linear elastic deformations of an isotropic homogeneous

material:

$$\begin{aligned} -\nabla\cdot\sigma &= f && \text{(force balance)} \\ \sigma &= \mathscr{C}:\varepsilon && \text{(constitutive equation)} \\ \varepsilon &= \frac{1}{2}(\nabla u+(\nabla u)^T) && \text{(strain)} \end{aligned} \tag{9}$$

where $\sigma, \varepsilon, u, f$ are the Cauchy stress and strain-rate tensor, displacement and body force, respectively.

The material parameter in the constitutive equation is given by the (fourth-order) stiffness tensor, $\mathscr{C}$, which serves as a linear map of the strain-rate tensor as $\mathscr{C}:\varepsilon=\sum_{k=1}^{3}\sum_{l=1}^{3}\mathscr{C}_{ijkl}\varepsilon_{kl}$. For a linearly elastic isotropic material

$$\mathscr{C}:\varepsilon=\frac{E}{1+\nu}\varepsilon+\frac{E\nu}{(1+\nu)(1-2\nu)}\operatorname{tr}(\varepsilon)I, \tag{10}$$

where E is Young's modulus and ν is the Poisson ratio of the material.

Additionally, when appropriate, we think of the TO problem that has been discretized using, e.g., the FEM leading to discrete representations $\{u_i\}_{i=1}^{n_u}$ and $\{m_i\}_{i=1}^{n_m}$ of the variables u and m, where each m_i and u_i is, for instance, a nodal or element value (degrees of freedom) of some basis function. A discretization of eq. (9) typically has the form

$$\mathbf{K}\mathbf{u}=\mathbf{f}, \tag{11}$$

where $\mathbf{K}$ is called the stiffness matrix, $\mathbf{u}=u_i$ is the discrete state solution vector and the load vector $\mathbf{f}$ is the discrete representation of f and, possibly, some boundary conditions. In the case of the FEM using the ansatz that the solution can be represented as a linear combination of basis functions ψ_i, i.e., $u=\sum_{i=1}^{n_u}u_i\psi_i$, the elements of the stiffness matrix are

$$\begin{aligned} \mathbf{K}_{ij} &= \int_\Omega\left[\frac{E}{1+\nu}\varepsilon(\psi_j):\varepsilon(\psi_i)+\frac{E\nu}{(1+\nu)(1-2\nu)}(\nabla\cdot\psi_j)(\nabla\cdot\psi_i)\right]\,dx \\ \mathbf{F}_i &= \int_\Omega f\cdot\psi_i\,dx+\int_{\partial\Omega_N}g_N\cdot\psi_i\,ds, \end{aligned}$$

where g_N represents an potential traction on a part of the boundary $\partial\Omega_N$.

When applied to design, Ω depends on m, and thus it is convenient to express that the stiffness matrix as well as the right-hand side in eq. (11) depend on the design parameter, thus rewriting the state equation as

$$\mathbf{K}(m)\mathbf{u}=\mathbf{f}(m). \tag{12}$$

We will, for simplicity, keep the same notation for the continuous variables (e.g, ρ, u) as for their discrete representations, given that this review does not focus explicitly on the discretization methods. The dependency of the state equation on m in the context of topology optimization is treated in chapter 2.

2 Methods in topology optimization

In this section we list some of the main approaches in TO and ongoing research, which has appeared since the homogenization method was introduced by Bendsøe & Kikuchi [6]. In addition to a number of extensive reviews, many authored by pioneers in the field of TO [e.g., 2, 19], the reader can find comprehensive reviews with specific focus on:

- Applications: structures [38–41], fluids [35, 42], acoustics [43], additive manufacturing [44–46], heat exchangers [47], nano-photonics [48] and fracture resistance [49]
- Numerical methods: immersed boundary methods [50], deep learning [51, 52] and isogeometric analysis [53].

There are also reviews specifically addressing the methods covered in this section: on level set methods [54] (section 2.2), explicit geometry TO [50, 55] (section 2.3), and gradient-free methods [56–58] (section 2.4, see table 4.2 for code availability from [57]). Theory and applications of density-based methods, introduced in section 2.1, are covered in the book by Bendsøe & Sigmund [1].

2.1 Density-based methods

When considering the design variable to be the material distribution, it should ideally only take two values, $m : \mathscr{D} \to \{0,1\}$, which divides the design domain into regions of solid material, $\{x \in \mathscr{D} : m(x) = 1\}$, and void, $\{x \in \mathscr{D} : m(x) = 0\}$. However, the original problem (3) and discretized versions of it, result in an ill-posed problem if imposing the constraint that $m = 0$ or 1. Refining the computational mesh typically gives patterns with alternating material distribution, resulting in a changing topology and non-convergence: this is referred to as *mesh dependence*. The original problem can be modified in a way so that a solution exists [59], by using *relaxation* (extending the space of admissible designs [6, 60]) and/or *restriction* (limiting the space of admissible designs to suppress high-frequency oscillations or material perforations [61, 62]) such as filtering.

Density-based Methods (DBM) introduce a pseudo *density* ρ of material in problem eq. (3), to replace the a priori binary-valued material distribution m. A basic TO problem can thus be formulated as

$$\begin{aligned} \min_{\rho \in \mathscr{F}_\rho} \quad & J(u(\rho), \rho) \\ \text{subject to:} \quad & F(u(\rho), \rho) = 0, \\ & g_0(\rho) = V_\rho - V_0 \leq 0 \end{aligned} \tag{13}$$

where

$$\begin{aligned} \mathscr{F}_\rho &= \{\rho(x) \,|\, 0 \leq \rho_{min} \leq \rho(x) \leq 1, \forall x \in \mathscr{D}\}, \\ V_\rho &= \int_{\mathscr{D}} \rho(x) dx. \end{aligned} \tag{14}$$

The minimum density ρ_{min} is often in practice set to a small positive value. Topology optimization using homogenization was initially developed in the 80s and early 90s [6, 63, 64]: the method considers materials constructed by infinitesimal and periodically repetitive microstructure that on the microscopic level, so called unit cells, consists of a homogeneous material with holes. The relaxation of the problem is then to extend the space of admissible designs to include all such composite materials. The variable density has a physical meaning in that it is a measure of the size of holes on the microscopic level and it is therefore necessary to relate ρ to the material properties. This is done using the homogenization procedure [e.g., 65, 66], which is an averaging of the state equations for the composite structure, and leads to similar PDEs as for linear elasticity but with modified material properties. In this

case the stiffness tensor $\mathscr{C}$ in eq. (9) becomes a so-called homogenized tensor dependent on ρ and the possible structure of $\mathscr{C}(\rho)$ should be added to the space of admissible designs $\mathscr{F}$ in eq. (13) [67]. Therefore, homogenization allows optimal solutions with well-defined intermediate values of ρ which describe the composite material. From a practical point of view, there often is a desire to have a structure consisting of solid material (i.e., composite material with with no holes on the microscopic level): a possibility is then to penalize intermediate values of ρ so that the final design converges towards a solid and void material [see e.g., 1]. Such a penalization process, however, does to some extent reduce the importance of the homogenized material tensor. This aspect of the homogenization method, combined with the non-trivial problem of finding closed expressions for the homogenized tensor for general composite materials, led to a modification of the method which penalizes regions with low stiffness in order to arrive at nearly binary density distributions [18]. This method is called Solid Isotropic Material with Penalization (SIMP) [27, 68, 69], and it has become a very popular method in TO.

2.1.1 Solid Isotropic Material with Penalization

In penalized density-based methods, ρ is in general related to the material properties by a material interpolation scheme, such as the SIMP [6] and the Rational Approximation of Material Properties (RAMP) [70] (for a comparison of material interpolation schemes, see Bendsøe & Sigmund [71]). The most common remain the original variation of the SIMP schemes [27, 72, 73]:

$$\mathscr{C}(\rho) = \rho^p \mathscr{C}_0 \quad \text{or} \quad \mathscr{C}(\rho) = \mathscr{C}_{min} + \rho^p(\mathscr{C}_0 - \mathscr{C}_{min}) \tag{15}$$

where $\mathscr{C}_0$ is the stiffness tensor representative of the solid material (e.g., the Young's modulus and Poisson ratio of the considered material), $\mathscr{C}_{min}$ is a tensor representing a very low-stiffness material and p is a penalty parameter, usually $p \geq 3$ [74]. This is equivalent, for a linear elastic material, to modifying the Young's modulus, E, in eq. (10) to for example $E_{min} + \rho^p(E_0 - E_{min})$. SIMP can be seen as a simplification of the homogenization method in that it circumvents the process of finding a homogenized tensor by assuming such a tensor is specified by a material interpolation scheme like eq. (15). Because of this the tensor described by eq. (15) does not describe a true microstructure material for intermediate density values and ρ is therefore often called a fictitious material [71, 75].

The principle of the penalty is to give advantage to the composition of regions with real material ($\rho(x) = 1$) or without material ($\rho(x) = \rho_{min}$), instead of gray zones ($\rho_{min} < \rho(x) < 1$). Observing that the amount of material is constrained ($g_0(\rho) \leq 0$ in eq. (13)), setting $p > 1$ makes it more economic to build structural strength based on regions of real material than on regions of gray material [18], i.e., a large enough penalization makes a region consisting of gray material disproportionately soft.

2.1.2 Restriction through filters

Even though the penalization encourages black-and-white designs, the SIMP approach in its basic formulation suffers from mesh dependency, such as checkerboard patterns or too jagged delineations between solid and void regions [59]. To prevent these effects and to ensure mesh independence and well-posedness, a restriction is applied through filtering techniques directly on the density [62, 74, 76] or on the sensitivities [59, 61]. Typically, a filter is implemented as the local weighted averages of the quantity. For example, a filtered density can be represented as:

$$\tilde{\rho}(x) = \int_{\mathbb{B}_R} w(x-y)\rho(y)\,dy, \tag{16}$$

where $\mathbb{B}_R$ is a ball of radius R and the filter function w is a positive function such that $\int_{\mathbb{B}_R} w(x)dx = 1$ [76]. The implementation of eq. (16) in a discretized form can be challenging and memory consuming, especially if the design domain is complex or if parallel

computations are to be performed. This motivated Lazarov & Sigmund [74] to proposed a PDE based density filter[4] as:

$$\begin{aligned} -r^2\Delta\tilde{\rho}+\tilde{\rho} &= \rho \quad \text{in } \mathscr{D} \\ \frac{\partial\tilde{\rho}}{\partial n} &= 0 \quad \text{on } \partial\mathscr{D}. \end{aligned} \tag{17}$$

Specifying the filter weight w in eq. (16) to be the Green's function $G(x,y)$ associated with eq. (17), the solution to the PDE fulfills $\tilde{\rho} = \int G(x,y)\rho(y)\,dy$. It was shown in [74] that the parameter r in eq. (16) is related through a constant scaling factor to R in eq. (16) when using a linear hat function as w. Since the filter is defined through the PDE, the same framework which is used to solve the state equation $\mathscr{F}(u,m)$ can be used to solve eq. (17), e.g., if the state solver is implemented using parallelism this can also be used for the computation of $\tilde{\rho}$. A density filter enforces the transition between solid and void regions to occur over at least a filter-radius length scale, effectively smoothing the density field and removing distributions of ρ that oscillate too much from the design space resulting in mesh-independent solutions. The filtering concept using a convolution or a PDE as in eqs. (16) and (17) can be applied to the sensitivity, instead of to ρ, making the updates of the design smoother. To produce a black-and-white design, and additional filter[5] can be applied to the (smoothed) filtered density [e.g., 77–79].

Other methods of ensuring mesh-independent solutions are formulations that explicitly control or penalize the perimeter or intermediate densities. Perimeter control methods regulate the q-norm of the gradient of the density by either adding $\|\nabla\rho\|_q$ as an additional constraint to the optimization problem or as a penalty factor to the objective function [80, 81], while regularized intermediate density control can be achieved by instead considering the functional $\int_\Omega \bar{\rho}(1-\bar{\rho})$, where $\bar{\rho}$ is either equal to ρ or a local average of it [e.g., 82, 83].

2.1.3 DBM summarized

DBM enables the utilization of efficient gradient-based optimization: it can introduce void or solid regions anywhere in the computational design domain $\mathscr{D}$, and the discretized state and adjoint-state equations can be implemented on a fixed mesh, which has computational benefits.

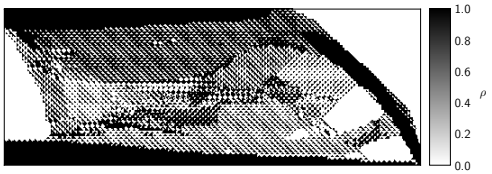
Two important challenges of DBM in applications are the slow convergence rates and the difficulty to apply non-natural boundary conditions on $\partial\Omega$; the latter limits the ability of the method to model physical scenarios as structures with specified loads at their boundaries [19], multiple physics which interact on $\partial\Omega$ [43] and problems considering point-wise stress constraints [37].

Figure 2.1 shows the result of a classical TO problem: the optimal distribution of a linearly elastic material that fulfills specifications representative of a bridge-like structure, called the Messerschmitt-Bolkow-Blohm beam, subject to a concentrated load in the top middle (fig. 2.1a). The cost to be minimized is the compliance of the structure, which is equivalent to maximizing the stiffness of the construction, under the constraint that the material occupies at most 50 % of the volume of $\mathscr{D}$. The SIMP method is used to formulate the TO problem and the OC method is used to solve the optimization problem. As can be seen in fig. 2.1b, using no filter results in a topology exhibiting a mesh-dependent checkerboard pattern. This is remedied in fig. 2.1c, where a filter without penalization, i.e., $p = 1$, has been used. This is specifically in 2D called the *variable-thickness-sheet* problem, since it generates areas of intermediate ρ values. In fig. 2.1d a typical value $p = 3$ has been used for the penalization

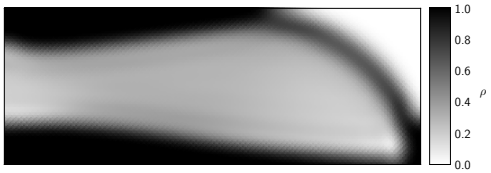
[4] This type of PDE based density filter is often called a Helmholtz type in the literature, even if the PDE is not the Helmholtz equation.

[5] These types of filters that mitigate gray zones in the topology and sharpen the solid-void transition in the design are often called projection filters in the literature.

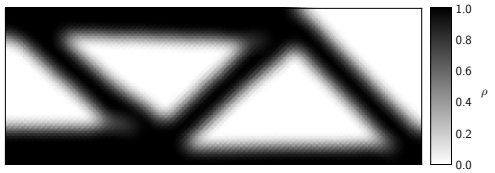
(a) Domain and boundary conditions.



(b) Result of solving the problem using the SIMP method without a filter.



(c) Result of solving the problem using the SIMP method with a filter and $p = 1$.



(d) Result of solving the problem using the SIMP method with a filter and $p = 3$.

Figure 2.1: The Messerschmitt-Bolkow-Blohm TO problem using SIMP. Only half the bridge structure is considered for the design domain $\mathscr{D}$ (light gray color), with symmetric boundary conditions on the left-hand side of the domain (no displacement in the x direction) and no vertical displacements at the lower right corner. The arrow indicates the applied force. The problem was solved using a script developed for this study, following the approach presented in [84, 85].

which, together with filtering, gives rise to a more distinct structure and ρ values that almost exclusively represent material or void.

2.2 Implicit boundary methods

Implicit boundary methods use some auxiliary function to represent the boundary of the material region in the design domain, $\partial\Omega$. This is the attractive feature of these methods since it generates designs with well-defined boundaries, in contrast to DBM where the boundary of Ω can be diffuse as it is defined as the frontier between material and void areas. The two main methods are the level set and phase-field methods.

2.2.1 Level set methods

The Level Set Method (LSM) is, together with SIMP, one of the widely used methods in TO. The method was initially developed for general moving-boundary problems by Osher & Sethian [86] and later used for TO [87, 88]. In this method the design, i.e., the limits of the regions with material, is parameterized using a so-called level set function ϕ (see fig. 2.2) of which the zero contour, typically, represents the boundary $\partial\Omega$:

$$\begin{aligned} \Omega &= \{x \in \mathbb{R}^n : \phi(x) > 0\} && \text{(material)} \\ \mathscr{D} \setminus \Omega &= \{x \in \mathbb{R}^n : \phi(x) < 0\} && \text{(void)} \\ \partial\Omega &= \{x \in \mathbb{R}^n : \phi(x) = 0\} && \text{(design boundary)} \end{aligned} \tag{18}$$

(some authors use opposite signs for the level set, e.g., $\phi < 0$ represents Ω). Compared to SIMP (section 2.1), the design parameterization using ϕ is commonly discretized independently from the state equation. Usual discretizations of the level set function are FE [e.g.,

89], Radial Basis Function (RBF) [e.g., 90, 91] and spectral functions [e.g., 92]. To compute the state equation for a given Ω, the geometry defined by $\partial\Omega$ implicitly through ϕ, which evolves during the optimization, needs to be mapped to the computational domain of the state equation. This mapping can typically be done in one of three ways: using a mesh for the state that conforms to $\partial\Omega$, using an ersatz (eng. *replacement*) material approach, or using a fixed grid and an immersed boundary method for the state equation.

With boundary-conforming mesh

The boundary *conforming mesh* approach has the benefit that the state equation is solved directly on Ω and that boundary conditions can be applied explicitly at $\partial\Omega$. Additionally, since the computational domain is limited to Ω, the cost of solving the state equation can be reduced compared to methods which require the solution of the state equations on the entire design domain $\mathscr{D}$ section 2.1.1. However, the mesh for the material domain must be modified at every update of ϕ (which changes $\partial\Omega$): this can be either be done by generating a new mesh for the state equation that has boundary $\partial\Omega = \{x \in \mathbb{R}^n : \phi(x) = 0\}$ [93] or modifying an existing mesh of $\mathscr{D}$, based on information of the evolution of the level set, so that a subset explicitly discretizes Ω (that is, the mesh is deformed, and new cells are possibly added, in a way such that there exists a set of cell facets that specify $\partial\Omega$) [94, 95].

With ersatz material

The approach using an ersatz material lets the level set function define a region of fictitious material in $\mathscr{D}$ [89, 96]. This is closely related to DBM in that the fictitious material can be expressed as a density function $\rho_{min} \leq \rho(\phi) \leq 1$, where $\rho(\phi < 1) = \rho_{min}$ and $\rho(\phi > 1) = 1$ in cells that are not cut by the zero contour. For cells that are cut by the zero contour some function, such as an approximate Heaviside function, is used to map ϕ to intermediate density values. Sigmund & Maute [19] discuss the similarity and overlap of the ersatz method and DBM such as SIMP. Like SIMP, a fixed mesh can be used for the state equation throughout the optimization process, but the application of boundary conditions on $\partial\Omega$ is challenging.

With immersed boundary methods

A possible cure to the ambiguity caused by introducing an ersatz material is to use an *Immersed Boundary Method* (IBM), such as Extended FEM (XFEM) [97–99] or cutFEM [100], which uses the level set function directly to define a boundary on a fixed computational mesh covering $\mathscr{D}$ [87, 101]. Boundary conditions for the state equation can be (weakly) applied at the level set and only cells where $\phi > 0$ and cut cells are used for the computation. Thus, this method to some extent combines the benefits of both the conforming mesh and the ersatz material approaches. It comes, however, at a cost: defining the domain of the state equation by $\partial\Omega = x \in \mathscr{D} : \phi(x) = 0$ relies on the IBM in the implementation of special quadrature rules to properly integrate over cut elements.

Calculating sensitivities

The material domain, and therefore the design, is defined by the level set function eq. (18). If the design problem, including the level set function, is discretized then the derivative of the cost function with respect to the discrete design variables m_i, i.e., $\frac{\mathrm{d}J}{\mathrm{d}m_i}$, is well defined. This is sometimes called a "discretize-then-optimize" approach [102]. On the contrary, if the design problem and the optimization process are defined prior to discretization, in an "optimize-then-discretize" approach, then the cost function is only defined with respect to Ω or $\partial\Omega$. Provided that $\partial\Omega$ is smooth enough, the *shape derivative* of the cost function can be defined as a Fréchet derivative $\frac{\mathrm{d}J}{\mathrm{d}\partial\Omega}$ and calculated [89].

In the case of the "discretize-then-optimize" approach and using a discretization $\sum_{i=1}^{n_m} l_i \tilde{\psi}_i$ of the level set function (where $\tilde{\psi}$ are basis functions), the design variables m_i are the parameters l_i, and the optimization problem can be solved using e.g., one of the algorithms

stated in section 1.3.1 [e.g., 97, 103, 104]. If the ersatz material method is used, the sensitivity can further, through the chain rule, be related to the (discrete) ersatz material as $\frac{\mathrm{d}J}{\mathrm{d}l_i} = \sum_e \frac{\partial J}{\partial \rho_e} \frac{\partial \rho_e}{\partial l_i}$, where the $\frac{\partial \rho_e}{\partial l_i}$ is the element-wise sensitivity of the material to the level set parameters. These will depend on the mapping used from the level set to the material domain. Alternatively, the evolution of the shape $\partial\Omega$ can be governed by a Hamilton-Jacobi type equation

$$\frac{\partial \phi}{\partial \tau} - v_n |\nabla \phi| = 0, \tag{19}$$

where τ is a pseudo-time and v_n is the design boundary velocity (i.e., v_n is the normal velocity at $\partial\Omega$). The optimization problem then becomes to solve eq. (19) until the level set function is steady. The design boundary velocity is connected to the design variables through an expression of the normal boundary variation [e.g., 102]. Using the "optimize-the-discretize" approach, v_n can be related to the shape derivative and selected in a way so that the evolution of the boundary is along $-\frac{dJ}{d\partial\Omega}$. Evolving $\partial\Omega$ through the Hamilton-Jacobi equation allows for topological changes through the merging of existing holes. Due to a maximum principle being fulfilled it however, regardless of which of the two approaches is used, does not allow for the emergence of new holes in the interior of the domain. For two dimensions, the only way a new hole can be created is by the boundary of an existing hole merging to a point so that a single hole is then separated into two. This process is unlikely in the case of maximizing stiffness [89]. In three dimensions this process is more likely since two pieces of the boundary can merge to "tunnel" a hole [105]. To mitigate the difficulty of creating new holes, topological sensitivity is often used as a part of the optimization process: for some optimization steps the so-called *topological derivative* [106, 107] is computed with respect to a topological variation, i.e., where it is most beneficial to introduce a small void in the interior of Ω. As a part of the update process the level set is then modified to include such a new hole.

Challenges and benefits

LSM share many challenges with DBM, e.g., well-posedness of the optimization problem (for instance mesh-dependent solutions) and possible lack of smoothness of the design Ω. Regularization methods (some similar or equivalent to those used in DBM) are used for the parameterization of the level set function to, for instance, control the gradients of ϕ. This promotes smoothness of the boundary [89] and, in the case of the ersatz method, penalizes intermediate values of ρ. If gradients become too small or too large as the level set evolves, it may be necessary to redefine ϕ to be the signed distance function to the current zero contour [87]. This procedure, called reinitialization, requires additional computations by, e.g., solving an auxiliary equation, and therefore increases the computational cost. Additionally, the Hamilton-Jacobi equation is numerically unstable and, if used, requires stabilization methods of some form such as artificial diffusion or a streamline/upwind scheme. Benefits are that the level set of ϕ defines a distinct boundary $\partial\Omega$ and that more general boundary conditions, like boundary-dependent loads, can be applied to this boundary. Figure 2.2 shows the results of using the LSM for the same TO problem as in fig. 2.1.

2.2.2 Phase-field methods

The Phase-Field Method (PFM) was originally developed for phase-transition problems by Fix [108, 109]. The PFM considers a phase-field variable $\psi(x)$ which approximates the indicator function of Ω in $\mathscr{D}$. Similar to the LSM the location of $\partial\Omega$ is determined by a level set function: here it is the level set of a pseudo-material density ρ, as the one defined in DBM, which is used to define the material domain boundary [110]. In the PFM, neither the boundary or the level set are explicitly followed in the optimization procedure [37].

We choose to follow the notations from Sigmund & Maute [19] and denote the phase-field variable by ρ, rather than the typical ψ, because, just like density, the phase field is defined on $\mathscr{D}$ and its values are in the interval $[0,1]$.

The method is based on minimizing a modified objective functional [19, 110], compared to original problem (3):

$$\bar{\mathscr{J}}(\rho) = \alpha\widetilde{J}(\rho) + \int_{\mathscr{D}} \left(\frac{1}{\varepsilon} W(\rho) + \varepsilon |\nabla\rho|^2 \right) \, dx, \tag{20}$$

where ε is the width of the transition zone between phases ($\rho = 0$ and $\rho = 1$) and $W(\rho)$ is a double-well potential satisfying $W(0) = W(1) = 0$, with α being a weight factor. The phase field (and thus $\partial\Omega$) evolves through solving one of the (pseudo) time-dependent Allen-Cahn or Cahn-Hilliard equations [111]. In this context $\bar{\mathscr{J}}$ can be seen as the free energy potential and the mass flux can be formulated using the sensitivity $\frac{\mathrm{d}\bar{\mathscr{J}}}{\mathrm{d}\rho}$. The evolution of the Allen-Cahn and Cahn-Hilliard equations can only result in a lowering of the free energy, i.e., $\bar{\mathscr{J}}$, and the optimization problem is solved by finding the asymptotic behavior of the solution ρ. Furthermore, the equations are volume preserving so the phase-field method incorporates a volume constraint like g_0 in problem (3).

In early applications of the phase-field method to TO, the volume constraint was implemented directly in favor of dropping the use of the Cahn-Hilliard equations, maybe due to the difficult nature of solving these numerically [112, 113]. As pointed out in Sigmund & Maute [19], in approaches minimizing $\bar{\mathscr{J}}$ directly, the distinction between phase-field methods and DBM is unclear when the latter explicitly penalizes gray material regions and large gradients, which are the roles of the terms $\frac{1}{\varepsilon}W(\rho)$ and $\varepsilon|\nabla\rho|^2$ in eq. (20), the latter acting as a sensitivity filter.

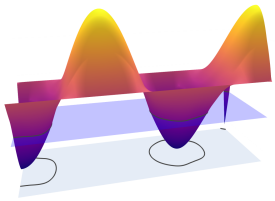
A downside of the method is that it often suffers from a slow convergence (slow-moving boundary in each design iteration) while a benefit is that, similar to LSM, it allows the application of loads on $\partial\Omega$ [112].

2.3 Explicit geometry methods

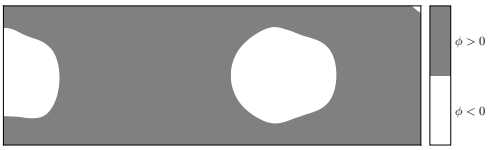
In the previous methods, the geometries of the design are deducted from material densities (section 2.1) or level sets of functions (section 2.2). On the contrary, when the geometry, or the shape of a component, is described explicitly by a function, the methods are herein called Explicit Geometry Methods (EGM). There are, to date, two approaches: the Geometry Projection Method (GPM) and Moving Morphable Components or Voids (MMC/MMV).

In the GPM, first proposed in [115], the explicit geometry of a structural component is projected onto a fictitious domain where, similar to previous methods described, the response analysis and sensitivity are computed. The method has the advantage that it can be used with both explicit (like stated above) and implicit geometry descriptions, and that it eliminates the necessity to penalize intermediate values (used in the fictitious domain) since the void-solid delineation is completely specified by the projection step with a precision level determined by the mesh resolution in the fictitious domain. However, GPM was initially a shape-optimization method rather than a TO method in that any holes in the domain had to be provided in the initial design guess (holes could not be deleted or added).

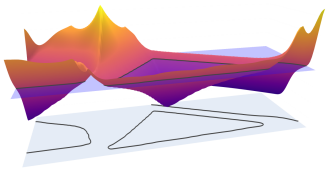
Guo *et al.* [116] introduced an EGM framework in which the geometry is described by several components which can change shape and position, including overlap. Each of the so-called moving morphable components has an initial rectangular shape and is described by explicit parameters. For the analysis, a level set function is used to describe the geometry (implicitly) and the XFEM [97] is used to compute the state equations. A big advantage of the MMC method is the ability to use splines and NURBS to describe the components, which simplifies eventual connections to Computer-Aided Design (CAD) representations of the geometry.



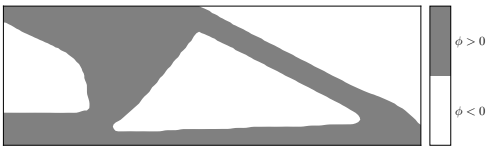
(a) Initial value of the level set function ϕ.



(b) Initial shape.



(c) Final value of the level set function ϕ.



(d) Final shape.

Figure 2.2: The Messerschmitt-Bolkow-Blohm TO problem using the LSM. In a) and c), the zero value is indicated by the light-blue plane. The contours where the zero value cuts the level set function ϕ are shown (projected onto the base level) in black. In b) and d) the corresponding contours delineate the solid ($\phi > 0$) and void ($\phi < 0$) material. The problem was solved using a modification of the code presented in [114].

A follow-up study [117] improved the method to account for 3D structures, components with variable thickness and to increase the computational efficiency.‡ Subsequent studies [118, 119] used B-splines to parameterize the components and introduced MMV, which explicitly describe the holes of a structure rather than the structure itself. The GPM method was in [120] adapted to account for, similarly to MMC, several geometrical structural components, parameterized by rectangular bars with semicircular ends which can overlap, to give an increased joint thickness, or be removed altogether. A `MATLAB` code using the GPM was later published in [121].‡

In the above methods, the design variables are the parameters describing the shape and position of the explicit geometry components. Since these tend to be fewer than the number of design variables used in DBM or IBM, gradient-free approaches can more often be used. However, both GPA and MMC/MMV are most commonly used with a gradient-based algorithm like MMA.

2.4 Gradient-free methods

In this section we briefly discuss the use of gradient-free optimization algorithms in TO. Broadly speaking, such methods are competitive, compared to gradient-based approaches, when the objective functional is not differentiable or for problems with a limited amount of design variables. A key strength of gradient-free methods is that they often, and in some cases can be proved to, converge to the global optimum. In addition, the algorithms are less sensitive to design-variable discontinuities. The downsides are that they require many evaluations of the design-problem functionals, cost and constraints, which becomes increasingly important as the number of design variables increase, and, in particular global methods, have a tendency to converge slowly [19]. For a comprehensive discussion on gradient-free and black-box optimization in a general context, see the book by Audet & Hare [122], where it is stated that from an efficiency point of view it is generally recommended to use gradient-based methods if the gradient information is reliable and available for a low computational cost.

Recently, Machine Learning (ML) approaches have contributed to improve the efficiency of gradient-free methods in TO [123]. Gradient-free optimization strategies which have been applied in TO are, among others, Bayesian optimization [124, 125], genetic algorithms [126, 127], swarm-intelligence methods such as particle swarm [128], ant colony [129] and bee colony [130] algorithms, and evolutionary algorithms [131].

TO-specific evolutionary optimization algorithms such as Evolutionary Structural Optimization (ESO, [132]) and Bi-directional ESO (BESO, [133]) seem to have reached a wider adoption in the TO community. The ESO method was initially developed as a gradient-free method that considers a design Ω, by removing material (binary-valued "density") elements, evolving towards an improved state and finally an optimum. The selection of elements to be removed, since it is not based on sensitivity information of the design, is based on heuristics. The method in its basic form is prone to checkerboard patterns in the design: such deficiencies have however been corrected [54]. Another drawback of ESO is that only the removal of material is considered, so the initial guess for the design must significantly overestimate the amount of material necessary for the optimal design: this was addressed by the BESO method which allows both adding and removing material. So-called "soft-kill" methods [e.g., 134] do not eliminate material but instead use a soft material. Such methods do, however, tend to incorporate sensitivity information and it is argued in Sigmund & Maute [19] that soft-kill methods are not gradient-free methods and that they closely overlap with the SIMP method.

‡see table 4.2 for code availability

3 Applications of topology optimization

This section focuses on research published on TO in the past decade with the aim to highlight current trends, and potential applications to airframe design, of the methods presented in chapter 2.

A comparison among commonly used TO methods, namely different versions of SIMP, BESO, and LSM, was made in [135]. Using `MATLAB` implementations[‡] of the methods, to minimize differences due to software, the study concluded that overall, for several test cases considered, the SIMP method using PDE or convolution filters gave the best results using the least computational resources. For a review on TO methods used in aircraft and aerospace design, with a particular focus on density-based methods, see Zhu *et al.* [136].

3.1 Aspects of geometry description

The typical structural optimization problem as stated in problem (3) with the simple state equation for an linear elastic solid (eq. (9)) has been the template for TO problems and is also the template for this section. The intent is to group by paragraph studies sharing the same TO approach (chapter 2). The section is finished with a paragraph on studies that in some way consider an aeronautical-related topic.

3.1.1 Isogeometric analysis

Recently, the use of Isogeometric Analysis (IGA, [137, 138]) in TO has gained particular attention. In contrast to traditional FEM the main idea of IGA is to use some form of spline, typically Non-Uniform Rational B-splines (NURBS), both for the basis functions and for definition of the computational mesh. In addition to having a higher-order representation of the geometry consistent with the representation of the solution of the discretized PDE, the use of NURBS allows a seamless connection between the discretization method and the CAD model. There are, however, some challenges in using standard IGA when considering complex geometries that include, e.g., holes, like those often occurring in TO. NURBS surfaces are is represented by the tensor product of NURBS curves and the domain used in the analysis should consist of a combination of patches of such surfaces, with each patch representing relatively simple shapes. In CAD models complex shapes are typically represented by so-called trimmed surfaces. Such surfaces are created by boolean operations of simpler geometrical objects, e.g., by defining a closed curve a hole can be introduced in a specific NURBS surface patch in the model. However, such a topological shape cannot be represented by a tensor-product NURBS surface in the analysis domain, and so several new patch typically have to be introduced that cover the surface surrounding the hole. This procedure results in an additional step and introduces computational complexity.

The earliest use of IGA in the context of TO was made in [139], where the authors addressed this topological inflexibility inherent in IGA by using so-called trimmed surface analysis [140] which directly uses the trimming information from the CAD model for the analysis. The method showed promise in that no parameterization of design variables or remeshing had to be performed but it, however, did suffer from computational complexity and inefficiency. Following studies expanded the use of IGA with simpler TO methods, like DBM [141], PFM [142], and the LSM [143]. More recently, [144–146] developed a method to use adaptive IGA with a phase-field model, which handles both mesh refinement and coarsening. As splines are used to discretize the geometry, even the coarse parts of the mesh can exactly represent the CAD geometry. The ability to both coarsen and refine thus increases computational efficiency without sacrificing accuracy.

[‡]see table 4.2 for code availability

In [147] a density-based IGA method is proposed where, to retain high smoothness and continuity requirements, a Density Distribution Function (DDF) is constructed using the same NURBS as the IGA. This leads to the reduction of wiggles in the design boundaries, and also increases the efficiency of the isogeometric TO. In a follow-up paper [148] the authors share a `MATLAB` code that implements the DDF method for compliance problems.[‡] Lambe & Czekanski [149] used a SIMP-based method with adaptive mesh refinement to get accurate boundary representations for complex geometries. The mesh adaptivity successfully suppresses numerical instabilities and unphysical phenomenons such as isolated spots of solid material. Wei *et al.* [150] succeed to combine the advantages of DBM and LSM, via a user-defined band parameter which specifies upper and lower bounds of the level set function improving the continuity of the objective and changes in topology without additional penalty or filtering parameters.

3.1.2 EGM applications

Hoang & Jang [151] extended the MMC method with the ability to impose constraints on both the minimum and maximum thickness of the bars. For the design of compliant mechanisms using hinge connections, the method suppresses the numerical instability typically present when using conventional density-based methods. As common in MMC TO, a benefit of the method is that the number of design variables is much lower compared to for instance SIMP, but a limitation is that components could only be removed during the design process and not added. The main challenge of going from the 2D test cases considered in the study to 3D was the efficient implementation of enforcing the increased number of local constraints that would be added. Along the same lines Zhang *et al.* [152] considered curved plates with the GPM with a motivation that such structures are readily available in commonly used mechanical structures. As an example it is shown that a bridge design consisting of single-curvature plates has a lower compliance given the same amount of material compared to using SIMP. Furthermore, the implementation allows for keeping the components within a specified, possibly non-convex, design envelope. Liu *et al.* [153] utilized the fact that the topology and state equations are fully decoupled in the MMC in order to propose what they called a multi-resolution approach, which uses different resolution meshes for the topology optimization and FE analysis. The multi-resolution MMC had the advantage of being computationally efficient, reducing the computational cost by an order of magnitude compared to similar multi-resolution TO methods using, e.g., SIMP, but had the limitation that the approach does not handle distributed loads well, with designs showing mesh dependency in this case.

3.1.3 IGA-enhanced EGM applications

The nature of MMC/MMV, that structural components or voids are parameterized by splines, integrates well with IGA, where splines are used for both defining the mesh elements and the basis functions. This union has lead to recent efforts to combine the methods for a TO framework better streamlined with CAD. Early work to combine the TO and analysis methods for 2D structural optimization was made in [154], where the authors showed that IGA resulted in greatly improved numerical stability compared to the original MMC-based TO. This can be attributed to the use of NURBS in IGA compared to linear FEM (the former being of higher order and accuracy). The method and implementation was extended to cover the 3D problem in Xie *et al.* [155] where the authors considered the problem of discontinuities due to overlapping structural components using MMC: areas where components intersect can be of low regularity and can even lead to a non-differentiable cost function. This problem was remedied by using functions that map the topology of overlapping components to a C^1 continuous representation. For this, however, a fine (NURBS) mesh was necessary resulting in high computational costs. This was improved in [156] by using a hierarchical local

[‡] see table 4.2 for code availability

(mesh) refinement. MMV and IGA were again combined in [157] to optimize the topology of complex shell structures. The trimmed surface analysis (used similarly as in [139]) was in this case informed by the morphable voids to introduce topological changes.

An extension of the soft-kill (gradient-based) BESO method was developed in [158] (the code is presented in a subsequent paper [159][‡]) where sequential programming was used to obtain a set of integer linear problems, the solution to each being a binary design. The proposed method showed an improvement compared to previous BESO approaches: convergence towards an optimum is improved without mesh refinement and general constraints can be treated explicitly, rather than what had previously been done with the BESO method which was to enforce constraints implicitly through the modification of J using Lagrange multipliers. Qiu *et al.* [160] use a BESO-like evolutionary method with a combined IGA/XFEM approach to produce a structure whose boundaries are specified by NURBS. Again, the use of NURBS is an advantage as the description of the geometry is directly compatible with CAD post processing, without losing accuracy or introducing errors, and eliminates the sometimes difficult step of going from a (typical) FE mesh of the design to a CAD-compatible geometry.

3.2 Potential of machine learning

3.2.1 ML-accelerated TO

Sosnovik & Oseledets [161] were early to apply a deep-learning approach to enhance structural TO problems solved by SIMP in order to converge to a binary distribution of material. The contribution of the ML approach, here a convolutional encoder-decoder neural network[‡], is similar to segmentation in image analysis: to identify regions based on pixel information. For the 2D compliance cases considered, a twenty-time improvement in computational efficiency was achieved when compared to directly solving the TO problem using SIMP. Large convolutional neural networks were also used in [162], also with training data generated using the SIMP approach. After training, TO problems with structural nonlinearities could efficiently be solved on less powerful hardware, e.g., a laptop, with high accuracy. Nie *et al.* [163] specify, in addition to the volume constraint, the loads and boundary conditions as inputs to a generative adversarial network, a generative/discriminative coupling of two neural networks[‡]. This is in contrast to previous deep-learning TO approaches that produce a map of the loads and boundary conditions to the finished design. The reasoning is that these physical fields serve as valuable information and will result in overall more accurate maps. The fully-trained model showed a speed-up of three orders of magnitude compared to a standard SIMP implementation for 2D test cases, with a zero-centered relative error of small variance of the volume fraction when compared to SIMP test cases. A ML approach, with training data created using structures optimized using the MMC framework, was applied in [164] to achieve real-time TO for 2D structures.

3.2.2 ML-driven TO

Whereas the applications cited above (section 3.2.1) demonstrate capabilities to improve the resolution of TO problems with neural networks trained on previous results, Chandrasekhar & Suresh [165] use neural networks to, through its associated weights and bias, represent a SIMP-based density field. This approach, in contrast to ML-accelerated approaches, thus directly uses the neural network as a part of solving the TO. By defining a loss function using this information, the problem (13) can be through a penalty formulation be turned into an unconstrained problem to be optimized. A side effect of this representation is that the density is independent of the underlying FEM mesh and that the resulting designs, in 2D and 3D, have crisp boundaries. The designs, however, tend to have less detail than the

[‡]see table 4.2 for code availability

corresponding designs optimized using SIMP directly, and a limitation is that the method tends to diverge for cases where distributed loads are used.

3.3 Giga-scale topology optimization

Aage *et al.* [8] used an efficient and highly parallelized SIMP method developed by the same team in [166][†] to compute the optimal design of an entire airplane wing. The study is a landmark in that the resolution of the computational (FEM) mesh is on the scale of millimeters for a wing-span of twenty-seven meters, resulting in designs that have giga-voxel resolution (voxel being a value associated with a 3D element in a regular grid). Excitingly, the optimized wing exhibits an internal structure with narrow-band regions similar to structural elements found in nature, e.g., bird beaks. Such a complicated structure may be viable to manufacture using additive manufacturing (see section 3.5) and, according to the authors, could save up to 5 % weight without lowering the stiffness significantly. Liu *et al.* [167][‡] presented an improved SIMP TO algorithm specifically developed to handle narrow-band regions like those found in the results of [8]. Using a block formulation of the underlying computational grid the method makes use of the design domain which consists mostly of void material and thus those parts can be excluded from the FEM. With a multi-grid preconditioner tailored for this block structure to make use of GPUs, the study demonstrated that designs with giga-voxel resolution could be produced on single workstations, as compared to the super-computer used in [8] (more than eight thousands CPUs during 1-5 days). Some of the TO test cases in the study consist of the design of a bird's beak and a generic airplane wing, which are similar and compared to the examples and results given [8].

3.4 Multidisciplinary aerodynamic design

A multidisciplinary approach was taken in [12] to optimize the landing gear of an airplane. A multi-body dynamic analysis was first performed on an initial design, which produced the dynamic loads on the landing gear. After this static loads that produce a similar response field as the dynamical loads were computed (using a method called Equivalent Static Loads, [e.g., 168, 169]) and used with a SIMP formulation to solve a TO problem. Depending on if the objective was to minimize compliance or manufacturing cost, the optimized designs showed significant weight savings and increase in peak stress (67 % and 74 %) with no cost savings, or a cost saving of 60 % (with weight savings and peak-stress increase at 36 % and 6 %, respectively).

Albanesi *et al.* [170] used shell (finite) elements to find wind turbine blades with a layout and topology optimized for minimizing the mass. They combine the use of a genetic algorithm, to determine the layout of the outer shell of the turbine and inner shear webs, with the subsequent use of SIMP to deduce a topology for the shear webs, and demonstrate weight savings of up to 28 % for a wind turbine blade. Träff *et al.* [171] present a computational framework for the TO of shell-like structures: the problem, formulated using SIMP and discretized using solid-shell elements, is solved using a geometric multigrid preconditioned Krylov method that is specifically adapted to be efficient for unstructured quadrilateral meshes. As a test case a simplified airplane fuselage, with an average mesh element size less than a centimeter, is considered for optimal reinforcement. Given a thin fuselage skin the resulting design exhibits a well-connected and dense reinforcement structure, void of too long and thin structural components, which improves resistance to buckling.

Wang *et al.* [172] developed a monolithic method (the coupled system is solved as a whole) to optimize the shape of a turbine blade represented as a thin-walled beam structure. The aerodynamic response was computed using the Blade Element momentum method, while

[†] see table 4.1 for code availability
[‡] see table 4.2 for code availability

a 2.5-dimensional beam FEM was used for the structural analysis. In the subsequent paper [173] the method was compared to two staggered methods, where the shape and topology are updated sequentially in an gradual manner, demonstrating that for most cases the monolithic method resulted in designs with lower compliance, which is attributed to the larger design space considered with the monolithic method. Both the shape and the internal structure of a full 3D wing are conjointly optimized in [174], with the goal of minimizing drag at cruise speeds subject to constraints on the compliance at takeoff and cruise conditions. A panel method is used to compute the aerodynamic response, which is one-way coupled to the linear elastic structural model. The results showed a significant reduction in both drag and wing weight (approximately 30 % and 40 %, respectively) when the design of the external shape of the wing was controlled by both local twist (angle) and chord, as compared to being controlled by only local twist.

3.5 Multiscale and multi- or anisotropic material structures

Additive Manufacturing (AM) enables the production of complex structures: multiple length scales, multiple interlaced materials, and heterogeneous materials exhibiting anisotropy. For recent reviews on the current state of AM, see e.g., [175–177], and [178] for a review focused on the aerospace industry. Combining TO and AM opens up for the production of high-performance structures in many fields, including for aeronautical use. With this possibility comes new challenges: the material of the design has the possibility of being constructed from layered laminates introducing orientation-specific (anisotropic or orthotropic) material properties, or similarly be made from multiple materials having different microstructure. Furthermore, AM can have the ability to produce closed-walled structures (built from variable-thickness sheets in 2D and shells in 3D), which have been shown to have a theoretically higher stiffness than truss- or frame-like structures for the compliance problem [179]; these types of designs have previously been discarded as unrealistic from a manufacturing perspective.

3.5.1 Advanced material modeling

SIMP, the most widely used method in TO is, in the form presented in section 2.1, a single-scale (homogeneous isotropic) material TO method and a simplification of the homogenization method. Using the terminology introduced in [179], open-walled structures (referring to structures consisting a lattice of line elements, e.g., a truss or a frame) typically result from using a penalization factor $p > 1$ in the SIMP (see fig. 2.1d), while unpenalized SIMP ($p = 1$) gives rise to closed-walled structures (structures which partially consist of sheet or plate elements, see fig. 2.1c). However, in some cases, for instance for a low material volume fraction, using a fine enough mesh with penalized SIMP the optimized design approaches a sheet or shell-like structure comparable to being closed-walled, indicating that the open-walled design in such cases are an inaccurate approximation of the true optimum [180].

Anisotropic materials

Zuo & Saitou [181] modify the SIMP method by using a material interpolation scheme that, by ordering the elastic modulus of materials by density, accounts for multiple materials in the design. The method, like regular SIMP, applies a penalty factor and therefore for the considered test cases gives open-walled designs with topology similar to previous studies. The possibility of combining any number of materials, which does not affect the computational cost of the method, does, however, open up the possibility for designs with a lower compliance. In a similar spirit, [182] consider multi-material TO: they approach the problem not by modifying the interpolation scheme, but rather use a so-called element-stacking

method which, by utilizing an element interpolation which considers the property of every cell of the design as a weighted sum of adjacent cells, effectively decouples the materials. The problem can the recast to a series of single-material problems that are solved using a SIMP method modified for anisotropic materials. Such an element-stacking method is used in Roper *et al.* [183] for a case study to simultaneously optimize for iso- and anisotropic multi-material distribution in an airplane seat.

Multiscale materials

Similarly to unpenalized SIMP, the homogenization method tends to produce a closed-wall structure when the problem approximates microstructure that is infinitely small. This quality together with the fact that the homogenization method is a multiscale material method, has resulted in a revived interest in the method for AM applications. However, there still exist some practical challenges in the manufacturing of truly multiscale materials directly relating to the optimal design, and an extra step is usually necessary to cast the design to a manufacturable product. *De-homogenization*, originally proposed in Pantz & Trabelsi [184, 185], refers to such a post-treatment process of going from the multiscale structure, the result of TO using homogenization on a typically coarse mesh, to a manufacturable structure by mapping this to a single-scale material with well-known mechanical properties on a fine mesh. The mapping typically considers the laminate orientation of the homogenized composite [186]. In [187] a formulation for single-scale representations of multiscale laminate materials in 2D that only exhibits a minor loss in performance is presented. This is extended to 3D in [188] where de-homogenization is used for the laminate material optimal designs given by homogenization for the linear elasticity compliance problem. The method, however, cannot cope with singularities (points at which the laminate direction is not integrable). The presence of singularities was investigated in [189], in which the authors present a singularity aware de-homogenization framework, enabling the introduction of local combinations of laminate directions to avoid the problems that a singularity can cause. Stutz *et al.* [190] use the homogenization method as a means to generate frame fields, which describe the orientations of laminates in the material, from which they produce a closed-walled multi-laminar 3D solid,[‡] through a collection of stream surfaces. An advantage of this de-homogenization process is that it is, unlike previous methods, unaffected by singularities. A similar approach is used in [191], where streamlines following the laminate orientation in the homogenization based optimal design are used to construct a graph, which is then used in producing the fine mesh used in the de-homogenization. The method treats singularities and distributed loads and has the potential to efficiently be implemented on GPUs, but is only 2D. Given that the mapping functions used in the de-homogenization process are not readily available and can be computationally intensive to solve for, Elingaard *et al.* [192] apply a deep-learning approach to reduce the computational cost of de-homogenization: by training convolutional neural networks to parameterize the mappings of lamination from the coarse mesh used for the homogenized TO problem to a fine mesh used for the final single-scale structure. Xu & Qian [193] similarly optimize for mapping functions in a two-step procedure: first the lattice orientation in the homogenized material is optimized, and second, a simultaneous optimization for compliance and mapping functions is performed, the latter of which are used in the de-homogenization process to obtain a single-scale graded lattice structure. Garnier *et al.* [194] apply a nonlinear reactive-diffusion equation to grow a lattice-like anisotropic pattern with a structure generated using SIMP in a pre-optimization step. For a classic TO problem considering a beam under multiple loads (Messerschmitt-Bolkow-Blohm beam, see figs. 2.1 and 2.2 for examples of single-load case), the method generates a structure that is able to support a 70 % higher peak load than the initial SIMP design. Furthermore, the method can be applied as a step in de-homogenization, where the application of the reaction-diffusion equation replaces the ML step used in a previous study by Elingaard *et al.* [192].

An approach different to the homogenization/de-homogenization optimization to generate

[‡]see table 4.2 for code availability

closed-wall structures is presented in [195], in which the authors use a two-step process consisting of first using SIMP TO on a fixed and coarse mesh and subsequently deforming the mesh to fit thin-walled features. The SIMP method uses a selective penalization designed to detect thin features of a scale possibly smaller than the mesh size which, for the designs considered in the study, produce closed-walled structures. The deformation algorithm, conceptually similar to solving a thermo-elasticity problem using intermediate density values in place of a temperature field, produces a body-fitted mesh (a mesh of the solid design) much easier to use for rebuilding the geometry using CAD.

3.5.2 Explicit geometry modeling

Also keeping the streamlining of TO and CAD in mind, Zhang *et al.* [196] used MMC where each of the morphable components were able to exhibit different material properties. This way a multi-material structure can be efficiently represented, given that the number of design variables is generally much lower using MMC compared to, e.g., SIMP or the homogenization method. The structural components are, on the other hand, typically larger than the scale represented using homogenization. The GPM was used to optimize the layout of a two-material periodic lattice structure in [197]. An adaptation of the homogenization method made it possible to impose necessary geometric restrictions, like symmetry, on the design of the unit cells. Kazemi *et al.* [198], however, pointed out the practical difficulty of extending the method proposed in [197] to more than two materials and presented an interpolation scheme developed for geometric components consisting of multiple (more than two) materials. Using this interpolation scheme, Kazemi *et al.* [199] proposed a modification of the GPM that uses homogenization to determine the unit cells of the material. The method has the capability to impose material symmetries (through the designed unit cells) and can therefore account for multi-material periodic lattice structures, and efficiently produces structures having high (maximized) bulk and shear moduli.

Looking for realizable designs

The overlapping of the structural components of different materials in the unit cells, however, pose a practical manufacturability problem: this could be solved by modifying the method to minimize overlaps and restrict overlaps to be of same-material components.

Several recent reviews have been published related to the topic of this section, indicating the high activity and fast evolution of both manufacturing and optimization methods. Osanov & Guest [200] discuss TO for periodic materials, Plocher & Panesar [201] give a view on design and optimization (TO and latticing) in AM mainly considering isotropic materials and Wu *et al.* [186] provide an extensive and up-to-date survey of TO in the field of multiscale materials and structures, as well as example `MATLAB` code for the structural TO of a rank-2 material[‡]. The focus of Ibhadode *et al.* [202] is on TO for metal AM and Bayat *et al.* [203] consider a holistic point of view for the combination of TO with multiphysics simulation AM processes in their comprehensive survey.

3.6 Fluid-structure interaction

Fluid-Structure Interaction (FSI) is a fundamental topic of aeronautics. This field is concerned with the mutual interaction between flow-induced forces and structural deformations.

DBM/SIMP

A potential difficulty for density-based methods, like SIMP, is that the typically non-smooth domains resulting 0-1 density distribution, i.e., a crisp design, have a significant impact on the flow field of the fluid. Lundgaard *et al.* [204] used carefully chosen interpolation functions to overcome such issues for a 2D FSI problem with steady incompressible

[‡]see table 4.2 for code availability

Navier-Stokes (NS) and linear elasticity. To reduce the nonlinearity that is added by the fluid-structure interaction, the structural analysis and sensitivity were computed only for the undeformed domain. In the aerodynamical setting, Gomes & Palacios [205] points out that common optimization objectives (e.g., minimizing drag or maximizing lift) may not benefit from a 0-1 design, leading to a not well-delineated design consisting of intermediate material density values. The study investigates the optimal material distribution in an elastic wing undergoing large deformations. The approach taken in the study to overcome the above challenges is to introduce two-step optimization process: first, the FSI problem is solved allowing for intermediate density values and, secondly, an inverse design problem is considered where a discrete (binary-like) topology that results in a matching response of the wing is produced. As test cases two different free-stream velocities, Mach numbers 0.25 and 0.5, are used with the objective to minimize drag under the constraints of generating enough lift and limiting deformation (avoiding buckling). Compared to other SIMP approaches commonly used to promote a distinct topology, e.g., increasing the penalty parameter or emphasizing stiffness in the optimization, this two-step approach shows improved convergence rates to a 0-1 design and thereby opens up for investigating the impact of different density filters and to what extent these are setting dependent. The same authors extended the study to include 3D wings at higher Reynolds numbers (corresponding to Mach 0.6) in [206].

Considering larger deformations, up to the point of brittle and ductile failure, Yoon [207] applied a DBM using a monolithic approach for the FSI problem developed by the same author in [208, 209] for stress-based TO. The proposed approach is monolithic in the sense that the governing equations and design variables are combined, using a special material interpolation function, requiring the solution of a unique system of equations defined on $\mathscr{D}$. Hence, a change in the design variable (density) affects the state equations directly. Yoon [210] further developed the monolithic approach for TO problems with transient FSI. The monolithic approach simplifies the transient sensitivity analysis compared to other FSI TO where the state equations are separated and coupled at the fluid-solid interface by boundary conditions, as in [e.g., 211, 212].

LSM

Jenkins & Maute [211, 212] developed a level set/XFEM method for the stationary incompressible NS flow around a linearly elastic object where the FSI problem is solved monolithically. The background mesh used for the analysis is deformable (the normal approach for an IBM would be to have a static fixed mesh) which, in combination with XFEM, allows for moderate deformations but avoids some of the more cumbersome reinitialization procedures of the level set function. The method allows for topological changes, but some difficulties with isolated solid parts and long slender structures were encountered in the test cases. To remedy this, the separated solid parts were tracked through an indicator function and modeled as part of the fluid flow throughout the optimization process.

Feppon *et al.* [213] considered the weakly coupled thermal FSI problem in 2D, where they solved the three-physics problem in a decoupled way by first solving the steady incompressible NS or Stokes equations in the fluid domain, then the convection-diffusion equation for the level set on the entire domain, and finally the linear elastic problem in the solid domain. The total decoupling of the equations is justified by assuming the small structural deformations would lead to negligible nonlinear effects in the coupled case. As a TO method, a level set method [214] that takes advantage of local mesh refinements was used. The refinement is formulated such that the zero contour of the level set is accurately represented by element boundaries, generating a sharp limit between fluid and structure in the FE. To compute the sensitivities a fully Lagrangian formulation is used, allowing for the treatment of general objective functions. The method was extended to 3D implementation, for which the refinement of unstructured meshes is challenging [32]. This approach of body-fitted meshes for the fluid-solid delineation was further developed in [215], where the use of the level set method with a reaction-diffusion equation and the ersatz method allowed for hole nucleation

(and merging) during the TO process.

BESO

Munk *et al.* [216] used the Lattice-Boltzmann equations and method to model the static flow of a fluid, coupled to FEM for the analysis of linear elasticity and the BESO method for the TO. This concept was first applied to micro-fluidic mixers where the objective was either to minimize the compliance of the mixer plate or to maximize the vorticity that the plate induces, with additional symmetry constraints on the structure [216], and was later extended to use SIMP and LSM [217]. The studies showed that for this type of FSI problem solving the fluid and structural state equations using a monolithic approach, while being more computationally expensive, generated a much better design (lower value of the objective function) when compared to solving the equations in decoupled manner. The CPU-based implementation of the method showed only minor computational improvements when compared to other methods, still making the computational runtime a bottleneck. In a subsequent paper, however, the authors leveraged the suitability inherent in the Lattice-Boltzmann method for a GPU-based implementation to improve computational times to be 1/20 of the similar CPU-based codes [218].

In a series of papers, Picelli *et al.* developed and applied a TO method of Binary Structures (TOBS, [158]) which, similar to DBM, uses gradient-based sensitivity analysis and integer programming, but only considers a 0-1 design variable (similar to BESO). For a low Reynolds number stationary incompressible NS flow the studies considered design-dependent loads [219, 220] and large deformations (geometric nonlinearity) [221]. The solution procedure uses an approach where the optimization is decoupled from the physics, enabling mesh refinement to resolve fine physical features, particularly around the fluid-structure interface, and still keep computational costs down. The physics are solved using an Arbitrary-Lagrangian-Eulerian approach, allowing for an accurate tracking of the fluid-structure interface. The study showed that including geometric nonlinearity results in significantly different designs when compared to linear elastic structures and that, in some cases, the design optimized for large displacements performed better (had a lower compliance) for both cases linear and nonlinear deformation. Therefore, including material nonlinearity is important to accurately optimize designs undergoing large deformations.

3.7 Radar cross section reduction

The shape and layout of structures and design of materials affects how they interact with electromagnetic waves. In aircraft design accounting for, or being able to some extent control, such interaction is an important part of low-observability technology. That is, the objective is to minimize the Radar Cross Section (RCS) of the aircraft. Approaches to passive RCS Reduction (RCSR) mainly consist of Shape Optimization (SO) of the aircraft and using radar-absorbent materials. More recently metasurfaces, thin (meta-) materials with structural patterns that are of sub-wavelength scale, have been explored as a way to scatter the reflected electromagnetic waves. With this in mind, we in this section focus on TO studies involving electromagnetic waves and have RCSR as an objective.

Even though this review does not explicitly cover SO, it can in the general sense be thought of as a subset of TO and we for informative purposes include what seems to be the fairly limited amount of studies considering SO for RCSR. In SO, the design variables are the parameters that describe the surface of an object, rather than the material distribution. The design variables are, for instance, the position of the nodes that define the surface mesh or shapes described by a set of parameters such as thickness, curvature, etc. The sensitivity used in gradient-based adjoint methods uses the *shape derivative*, which expresses the variation of the objective resulting from a displacement of the shape (surface). A common design requirement to reduce the RCS is to obtain a shape which for an incoming radar wave

minimizes the wave reflection back in the same direction. A gradient-based "optimize-then-discretize" approach is used in [222], where the sensitivity is given by solving the scattering problem for an incident wave in 2D. To avoid wave-length dependent corrugation-like oscillations in the designs and to produce smooth shapes, a penalty parameter has to be added to the objective function. The resulting shapes exhibit sharp tips in the direction that the RCS is minimized. The study also considers, by adding the NS equations to the state, a weighted objective that combines RCSR and the reduction of aerodynamical drag and finds that shapes giving a low RCS increase the drag. The method developed is later used in [223], where the optimization of cylindrical metal structures are examined. Such structure are commonly used as struts in various components, e.g., reflector antennas, and their scattering properties can be an important part of the overall RCS. Similarly to [222], the study found that asymmetrical shapes elongated in the direction of the incoming wave resulted in a lower RCS.

An approach that used a "discretize-then-optimize" method was developed in [224] for combined aerodynamic and RCSR optimization. The study finds that, similar to [222], the optimal design resulting from aerodynamic-RCSR optimization had far inferior RCS values compared to the optimal design resulting from the objective considering only RCSR. However, the aerodynamic performance of the combined optimization design did not suffer significantly when compared to only aerodynamic optimized design. In the same vein, [225, 226] used gradient-based optimization with combined aerodynamic-RCSR objective to find optimal designs for a flying-wing aircraft and an unmanned aerial vehicle. The findings regarding design performance are comparable to those of [224] in that trade-offs are made between aerodynamic and RCSR capabilities and that these are determined by the weights of each in the objective function. The case where the surface of the object (e.g., aircraft) is subjected to an external load was considered in [15]. The objective was to minimize the RCS variation due to structural deformation of the surface which was accomplished using SIMP TO of a planar reinforcement structure located under the surface. The method and results of the study serve as a complement to other studies focusing on RCSR using SO, since the basic shape of the structure is preserved (rather than changed) and optimized to minimally affect the RCS under small changes in shape due to deformation.

Despite its applicability for practical RCSR, SO is not without its shortcomings. Like stated above, changing the shape of an aircraft for RCSR typically reduces its aerodynamical performance. Additionally, the shapes are often optimized for a narrow band of frequencies. The use of metasurfaces (MS) has the potential to address both of these issues in that they do not change the shape significantly (only the surface) of an aircraft and can, possibly, be manufactured to scatter radar waves over a broader span of frequencies. Several recent reviews on the advancements in the field of MS have been published and, in particular, the following are relevant to the topic of this section: reviews on MS for RCSR [227], low-observability applications [228], as absorbers [229] and most recently as broadband absorbers [230]. A shorter paper discussing MS design based on TO is provided by Fan [231].

In Haji-Ahmadi *et al.* [232] a particle-swarm optimization was used to design a chessboard-like metasurface consisting of two kinds of artificial magnetic conductors, making up the unit cells of the surface. The optimization procedure consisted of determining the layout of the two unit cells so that the phase difference of the reflection of an incoming wave are as close to 180° as possible, which would make the waves cancel out. A frequency-domain electromagnetic solver with periodic boundary conditions was used for the state equation and to compute the objective function. The optimal design was numerically shown to induce a phase difference of $(180 \pm 37)^\circ$, resulting in a >10 dB RCSR over a wide band of frequencies (3.6 GHz – 10.4 GHz) and for incidence angles up to 40°. Additionally, an experimental setup using a manufactured MS based on the simulations validated the numerical results. The same general approach was followed in [233] to design a radar-absorbing unit cell layout with the objective to maximize the absorptivity. The numerical output showed

a significant ($>$10 dB) RCSR, but over a relatively narrow band (2.8 GHz – 3.5 GHz). As in [232], the numerical experiments were validated by tests on a manufactured MS. Min *et al.* [234] applied a genetic algorithm with the same electromagnetic solver as in previous studies and demonstrated that the general method worked well for the design of single unit-cell flexible transparent metamaterial absorbers. Also this study confirmed the results experimentally, showing a $\geq$10 dB RCSR at 10 GHz for bending angles up to 45° of the unit cell and a 90 % absorptance over the frequency range 5.3-15 GHz.

4 Software

In this section we list some of the TO codes that are freely available and that have been referenced in the literature review. The explicit requirement on the software listed is that the (numerical) implementation of the specific TO method should, for educational purposes, be open, i.e., that the code implementing the TO method is available. The numerical implementation for the state equation (e.g., eq. (9) using FEM) does not necessarily have to be open and the underlying numerical framework does not have to be open source (such as, and in particular, `MATLAB` which is often used for educational codes in the TO community), even though we have made an effort to mainly include codes that show the complete implementation, at least in the given framework. We subdivide the types of software into general-purpose software (section 4.1, code availability summarized in table 4.1) and problem-specific codes which supplement a particular paper or specific scenario (section 4.2, code availability summarized in table 4.2).

A recent, and extensive, review that focuses on open-source TO codes is given in [235], where codes are categorized by TO method and ranked by ease of use. Additionally, as a part of the study by [236] (which also documents a LSM TO package, `ParaLeSTO`) a smaller list of open-source TO software is provided.

In the following sections short descriptions will be provided of the studies for a particular software/code if the study has not already been cited in the paper.

4.1 General-purpose software with TO

General-purpose software, as presented here, are as a rule (a collection of) libraries with a unified API simplifying the setup necessary to numerically solve a specific problem, generally represented by a PDE. The software may have access to, e.g., computing the adjoint of a PDE, automatic or symbolic differentiation, shape derivative and sensitivity, which can be used to streamline the implementation of some of the (commonly used) TO presented in chapter 2. We here only include general-purpose software that to some extent have documented the use of TO specifically and that, at least in principle, have the possibility to be readily adjusted through changing, e.g., the design domain, state equation or objective function.

`FEniCS` [237, 238] is a frequently used platform to solve general PDEs using the FEM. A domain-specific language using a (high-level) symbolic-like syntax simplifies the definition of the physical/mathematical problems. Leveraging the ability in `FEniCS` to compute derivatives and adjoints of complex expressions, a general framework for PDE-constrained optimization, called `dolfin-adjoint` (more recently the library has change its name to `pyadjoint`) was implemented as an add-on package in [17, 239]. In particular, `dolfin-adjoint` works by overloading much of the `FEniCS` functionality: a benefit is that only the forward problem needs to be stated, with the adjoint system automatically derived from this, while a drawback is the black-box nature of the optimization procedure this introduces. Demos[6] show density-based TO for Stokes-flow and the heat equation as state equations. Alonso *et al.* [240] consider fluid TO using `FEniCS` with `dolfin-adjoint` coupled to `OpenFOAM` [241]. Even though optimization of fluid flow is not the topic of this literature review, this software is included for (potential) educational purposes particularly given that several TO software build upon the `FEniCS` framework. A more specific software using density-based TO was presented in [242], where the authors couple `FEniCS` (used for the PDEs) and `OpenMDAO` [243], a framework for multidisciplinary design using gradient-based optimization. Several state equations (linear and non-linear elasticity and thermoelasticity)

[6] `https://www.dolfin-adjoint.org/en/stable/documentation/examples.html`. Accessed 2026-03-09.

and SIMP/RAMP are implemented, and there are documented cases.[7] A "pure" FEniCS implementation was given in [244], where a parallelized 55-line code for structural SIMP TO is provided; through the versatility of the FEniCS framework the potential exists for the code to be extended by, e.g., the type of state equation, with relatively small effort.

NGSolve [245] is a multi-physics FE software, with an interface similar to FEniCS, that has the capacity to use symbolic differentiation facilitating gradient-based SO and TO: documented demos exist using the LSM and topological derivative for 2D structural problems. The software also supports IBM, even though this has not explicitly been used in the SO/TO problems. A similar approach of high-level symbolic syntax is used in Gridap [246], a tool for solving general PDEs written in the julia language. Here, again, the generality/flexibility of the software is demonstrated by implementing SIMP (with FEM) for an electromagnetical TO problem[8]. Like NGSolve, Gridap has support for IBM, but this is separated from the TO implementation. The FE framework MOOSE [247] has as part of its combined-module functionality implemented SIMP TO examples using various filters, mesh adaptivity, multi-load and multimaterial, for structural TO, and an example of a thermo-mechanically coupled problem.[9]

Developed at CIMNE by À. Ferrer and colleagues, Swan[10] is a structural TO software developed in MATLAB that has density-based, phase-field and level set methods implemented. The software is general in the sense that several methods and constraints are supported to cover a wide variety of structural TO problems, but it is not currently clear (to the authors) to what degree it is extendable to cover, e.g., other (multi-) physical models or numerical methods. The documentation of Swan has many tutorials for various structural TO problems.

4.2 Problem specific TO codes

Sigmund [84] early on advocated for the reproducibility of results through sharing codes used to generate test cases in studies and set an example for this in the paper which presented a 99-line educational MATLAB code for 2D structural SIMP TO. Since then, Sigmund and colleagues have published several papers on the same topic or included available code in studies. This has to a degree set a trend in the TO community as many other studies have followed suit, by either providing the code as an appendix in the paper, through a public website, or through one of the major collaborative version controlled developer platforms like www.github.com. This is exemplified by the TopOpt group (O. Sigmund) that hosts their software and codes, focusing on density-based TO, at their website and include, typically MATLAB, code in the many educational papers they published. G. Paulino's team has published several educational papers in which they gradually evolve a density-based TO method that uses the Virtual Element Method (VEM, FEM extended to used arbitrary polygonal meshes), which they show suppresses numerical instabilities, such as checkerboard patterns and isolated small-scale patterns. In a series of papers they address TO for 3D (PolyTop3D [248]), multi-material (PolyMat [249]), stress-constrained (PolyStress [250]) and dynamical-load (PolyDyna [249]) structural problems.

Following in the tradition of the 99-line [84] and 88-line [85] educational SIMP TO codes, [251] presented a code extended to 3D that is more efficient. Träff *et al.* [252] implemented a GPU-accelerated SIMP TO code for linearly elastic structural problems. In the same vein, [253] provided a MATLAB LSM code for similar, but 2D, TO problems.

Laurain [114] implemented an educational code using the FEniCS framework where they considered a two-phase material (a weak and a strong phase) and used the distributed (or

[7] https://lsdolab.github.io/atomics. Accessed 2026-03-09.

[8] https://gridap.github.io/Tutorials/stable/pages/t019_TopOptEMFocus. Accessed 2026-03-09.

[9] https://mooseframework.inl.gov/modules/combined/. Accessed 2026-03-09.

[10] https://swanlab.github.io/Swan. Accessed 2026-03-09.

volumetric) shape derivative [254], rather than the commonly used boundary shape derivative, to compute the sensitivity. The distributed shape derivative, which is computed over the domain, eliminates the need to specify a normal boundary velocity like v_n in eq. (19), and instead uses a more general Hamilton-Jacobi equation to evolve the level set function. Laurain [114] point to that the implementation of a distributed derivative is simpler than that of the boundary shape derivative and can result in more accurate [255]. Furthermore, they state that a benefit of not having to compute v_n is that this may introduce an additional interpolation error when the level set is not defined strictly at grid points, and that multiple interpolations have to be used when when the there is a jump in v_n, which is the case at the interface between the two phases. Figure 2.2 shows results using (a slightly modified version of) the code presented in [114].

Table 4.1: General-purpose software with TO software. Abbreviations used in the table: [gh] (https://github.com), [to] (https://www.topopt.mek.dtu.dk). All webpages accessed on 2026-03-09. See Nomenclature for other abbreviations.

Reference	TO (numerical) method	Physical model	Framework	Availability
[256]	MMC (FEM/FDM)	Structural	MATLAB	[gh]/jnorato/GPTO
TopOpt_in_PETSc [166, 257]	SIMP/Homogen. (FEM)	Structural (extendable)	PETSc, C++, Python	[gh]/topopt/TopOpt_in_PETSc [gh]/thsmit/TopOpt_in_PETSc_wrapped_in_Python
dolfin-adjoint [17, 239]	SIMP as demo (FEM)	General (extendable)	FEniCS, Python	[gh]/dolfin-adjoint/pyadjoint.git
[240]	SIMP (FEM/FVM)	Fluid	dolfin-adjoint, OpenFOAM	[gh]/diego-hayashi/fenics_topopt_foam
topo-fenics [244]	SIMP (FEM)	Structural (extendable)	FEniCS, Python	[gh]/iitrabhi/topo-fenics
NGSolve	LSM as demo (FEM)	General (extendable)	C++, Python	[gh]/NGSolve/ngsolve
Gridap [246, 258]	SIMP as demo (FEM)	General (extendable)	C++, julia	[gh]/gridap/Gridap.jlgridap/Gridap.jl
Swan	SIMP/LSM/Phase-field	Structural	MATLAB	[gh]/SwanLab/Swan

Table 4.2: Problem specific TO codes. All codes consider the objective to minimize compliance. Abbreviations used in the table: [gh] (https://github.com), [to] (https://www.topopt.mek.dtu.dk). All webpages accessed on 2026-03-09. See Nomenclature for other abbreviations.

Reference	TO (numerical) method	Physical model	Framework	Availability
topRank2, [186], [85]	SIMP	rank-2 laminate microstructures with single load	MATLAB	Paper app.
top99neo, top3D125 [251]	SIMP	Structural with multiple loads	MATLAB	[to]/apps-and-software
[190]	Homogen.	Multi-laminar structure with multiple loads	C++, MATLAB	[to]/apps-and-software
[252]	SIMP	Structural with multiple loads	GPU (OpenMP or Futhark)	[gh]/topopt
IgaTop, [148]	DDF (IGA)	Structural with single load	MATLAB	Paper app.
[57]	BESO	Structural with single load	MATLAB	Paper app.
[121]	GPM	Structural with single load	MATLAB	[gh]/jnorato/GPTO
[117]	MMC	Short-beam problem	MATLAB	Paper app.
[253]	LSM	Structural with multiple loads (2D)	MATLAB	Paper app.
[159]	TOBS	Structural with single (2D)	MATLAB	Paper app.
[114]	LSM	Structural (2D)	FEniCS	antoinelaurain.com/compliance.htm
OpenLSTO [236]	LSM	Structural (3D)	FEniCS, C++	[gh]/M2DOLab/OpenLSTO
Poly[Top3D] [Mat] [Stress] [Dyna] [248–250, 259]	SIMP/RAMP (VEM)	Structural (2D or 3D)	MATLAB	paulino.princeton.edu/software.html
nn4topopt [161]	SIMP (ML and FEM)	Structural	Python	[gh]/ISosnovik/nn4topopt
spgrid_topo_opt [167]	SIMP	Structural	C++, CUDA	[gh]/yuanming-hu/spgrid_topo_opt
TopologyGAN [163]	SIMP	Structural	Python, TensorFlow	[gh]/zhenguonie/2020_TopologyGAN

5 Summary and outlook

This document presents an overview of TO methods: methods for the optimal design of structures based on the topological description of their material layout and constrained by the physics involved in the design problem, e.g., a cost function to minimize and additional constraints coming from specifications. In this setting, the *design variables* control the distribution of material in a predefined domain where the structure will be designed, which essentially is the only guess that the designer needs to make when defining the optimization process.

We give an introduction to the fundamental concepts and methods which serves as a basis for the review of recently published TO literature, with a focus on fields relevant for aeronautical applications. The methods are categorized as *gradient-based*, such as DBM (SIMP and homogenization), LSM, PFM and MMC, or *gradient-free*, such as evolutionary algorithms ESO and BESO. Gradient-based methods make use of the sensitivities, i.e., how the objective and constraints change with perturbations in the design parameters. The sensitivities can be efficiently computed using the adjoint method when the number of design parameters is larger than the number of constraints. The adjoint approach has been dominant in TO, but gradient-free methods, which are not as computationally efficient, have recently gained importance through the use of machine learning.

This literature review indicates that the adjoint method still dominates, but the prevalence of a specific type of TO methods varies depending on the type of application. For the optimization of elastic structures IGA is often used and combined with some adapted version of the SIMP method. Another direction is to improve the efficiency of TO methods to be able to consider designs consisting of billions of design parameters: for such high-performance computations the SIMP method is most regularly employed. When considering structures consisting of multiscale or heterogeneous materials, the SIMP method often appears to be a too severe simplification to use on its own. Instead, homogenization is used due to its ability to properly account for multiple length scales. This method, that laid the basis for numerical TO when it was introduced in the late 1980s, fell out of flavor due to its rather complex formulation. For multi-material structures EGM, which consider the arrangement of individual structural components rather than a density field, have been successfully adapted. Topology optimization in FSI is highly dependent on that the boundary between the phases is well resolved for the well-posedness of the coupled state equations. As such, a good approach are methods that use a level set method in combination with either an immersed boundary method or some other accurate representation of the boundary through mesh refinement or body fitting. Finally, TO for RCSR appears to have only a limited amount of recent studies published. The direction in this area seems to be towards the use of designing MS, since these potentially have the ability to reduce the wave scattering over a broad spectrum of wave lengths. TO for MS considers only a limited amount of design variables, which makes the use of gradient-free methods well-suited.

The final section of this document lists software and codes that implement various TO methods in either their basic form or directly related to published papers. With codes that are fairly short, the prospect is to increase the impact of the publications, in education, research or in engineering, with possible re-implementations in general numerical framework, e.g., `FEniCS`, `NGSolve` or `GridAp`.

Future work may include expanding the script developed in this study (implemented using `FEniCS` and used to produce fig. 2.1) to take into account more general TO problems and implementing more specific TO methods, depending on application, presented in some of the literature cited in this study.

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